Essence of Geometric Algebra

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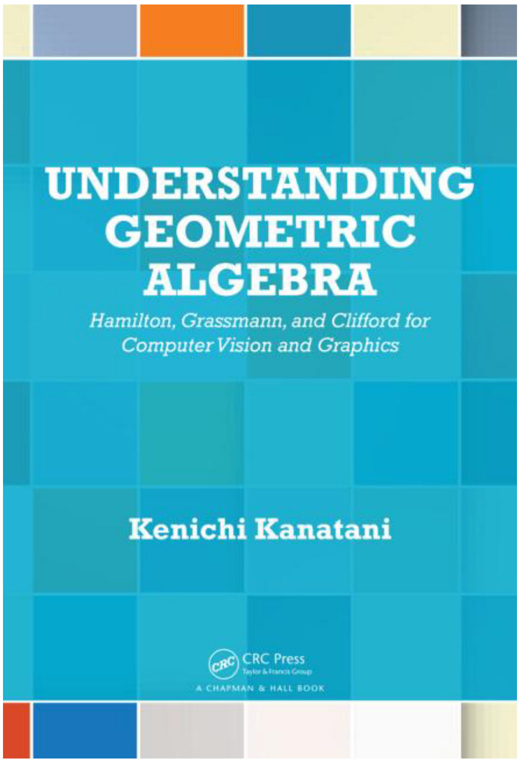

UNDERSTANDING GEOMETRIC ALGEBRA
Hamilton, Grassmann, and Clifford for Computer Vision and Graphics
Kenichi Kanatani
CRC Press
Taylor & Francis Group
A CHAPMAN & HALL BOOK

Introduction

What is geometric algebra?

Geometric algebra ≈ Clifford's algebra
= Hamilton's algebra + Grassmann's algebra

coupled with

Grassmann–Cayley algebra (≈ projective geometry)
+ conformal geometry.

Historical developments

vector calculus	$\longleftarrow$ Gibbs	Hamilton's algebra Grassmann's algebra	$\longrightarrow$ Clifford	Clifford's algebra

simpler $\longleftarrow$ $\longrightarrow$ more general

Algebras

An *algebra* is a set of elements closed under addition, scalar multiplication, and products:

algebra = vector space + products.

- The products must be associative: $(AB)C = A(BC)$.
  - They need not be commutative: $AB \neq BA$.
- The products are linearly distributive to the addition and scalar multiplication:

$$\alpha(A + B) = \alpha A + \alpha B, \qquad (\alpha + \beta)A = \alpha A + \beta A.$$

- Examples of algebras:
  - The set of real numbers $\mathbb{R}$
  - The set of complex numbers $\mathbb{C}$
  - The set of polynomials
  - The set of $n \times n$ matrices $GL(n)$

Formal sum

We can add *anything*: the sum is merely a *set*.

$$\text{oranges} + \text{apples} = \{\text{oranges}, \text{apples}\}.$$

The usual summation rules are applied:

$$(2\,\text{oranges} + 3\,\text{apples}) + 3\,\text{oranges} = 5\,\text{oranges} + 3\,\text{apples}.$$

Hamilton's quaternion algebra

Sir William Rowan Hamilton (1805–1865)

Quaternions

- Consider an algebra *generated* by 1 and symbols i, j, and k
  (= the smallest algebra that contains 1, i, j, and k).
- We require that the product (*quaternion product*) be subject to the rule

$$i^2 = j^2 = k^2 = -1,$$

$$ij = k, \qquad jk = i, \qquad ki = j,$$

$$ji = -ij, \qquad kj = -jk, \qquad ik = -ki.$$

  - The quaternion product is not commutative.

	1	i	j	k
1	1	i	j	k
i	i	-1	k	$-j$
j	j	$-k$	-1	i
k	k	j	$-i$	-1

- Each member (*quaternion*) q of this algebra is a formal sum in the form

$$q = \alpha + \beta i + \gamma j + \delta k$$

cf. The set of complex numbers is an algebra generated by 1 and a symbol i with the product rule

$$i^2 = -1.$$

  - Each member z (complex number) is a formal sum in the form $z = \alpha + i\beta$.
  - The set of complex numbers is a *subalgebra* of the quaternion algebra.

	1	i
1	1	i
i	i	-1

Vectors and quaternions

- If a 3-D vector $\boldsymbol{a} = a_1 e_1 + a_2 e_2 + a_3 e_3$ is identified with the quaternion $\boldsymbol{a} = a_1 i + a_2 j + a_3 k$, a quaternion can be expressed in the form

$$\begin{aligned} q &= \underbrace{\alpha}_{\text{scalarpart}} + \underbrace{\beta i + \gamma j + \delta k}_{\text{vectorpart}} \\ &= \alpha + \boldsymbol{a}. \end{aligned}$$

- The product of two quaternions $q = \alpha + \boldsymbol{a}$ and $q' = \beta + \boldsymbol{b}$ has the form

$$qq' = (\alpha\beta - \langle \boldsymbol{a}, \boldsymbol{b} \rangle) + \alpha \boldsymbol{b} + \beta \boldsymbol{a} + \boldsymbol{a} \times \boldsymbol{b}$$

$\langle \boldsymbol{a}, \boldsymbol{b} \rangle$: the inner product of $\boldsymbol{a}$ and $\boldsymbol{b}$ $\qquad$ $\boldsymbol{a} \times \boldsymbol{b}$: the vector product of $\boldsymbol{a}$ and $\boldsymbol{b}$

- The quaternion product of two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is

$$\boldsymbol{ab} = -\langle \boldsymbol{a}, \boldsymbol{b} \rangle + \boldsymbol{a} \times \boldsymbol{b}.$$

  - *The quaternion product computes the inner product* $\langle \boldsymbol{a}, \boldsymbol{b} \rangle$ *and the vector product* $\boldsymbol{a} \times \boldsymbol{b}$ *at the same time.*

Conjugate and inverse

- The *conjugate* of a quaternion $q = \alpha + \boldsymbol{a}$ is defined to be

$$q^\dagger = \alpha - \boldsymbol{a}.$$

- Then, we have

$$qq^\dagger = q^\dagger q = \alpha^2 + \|\boldsymbol{a}\|^2 \;\; (\equiv \|q\|^2)$$

- This means

$$q\frac{q^\dagger}{\|q\|^2} = \frac{q^\dagger}{\|q\|^2}q = 1.$$

  Hence, every q $(\neq 0)$ has its *inverse*:

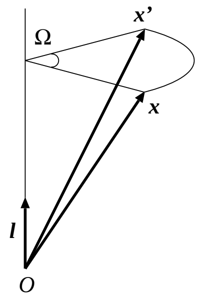
$$q^{-1} = \frac{q^\dagger}{\|q\|^2}, \qquad qq^{-1} = q^{-1}q = 1.$$

  - The set of all quaternions is a *field*.

Rotation

- Suppose a vector $\boldsymbol{x} = xi + yj + zk$ is rotated around $\boldsymbol{l}$ (unit vector) by angle Ω to $\boldsymbol{x}' = x'i + y'j + z'k$. Then,

$$\boldsymbol{x}' = q\boldsymbol{x}q^\dagger, \qquad q \equiv \cos\frac{\Omega}{2} + \boldsymbol{l}\sin\frac{\Omega}{2}.$$



- In terms of the axis $\boldsymbol{l}$ and the angle Ω, we obtain *Rodrigues's formula*:

$$\boldsymbol{x}' = \boldsymbol{x}\cos\Omega + \boldsymbol{l}\times\boldsymbol{x}\sin\Omega + \langle\boldsymbol{x},\boldsymbol{l}\rangle\boldsymbol{l}(1-\cos\Omega).$$

- In matrix form, we can write

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} q_0^2+q_1^2-q_2^2-q_3^2 & 2(q_1q_2-q_0q_3) & 2(q_1q_3+q_0q_2) \\ 2(q_2q_1+q_0q_3) & q_0^2-q_1^2+q_2^2-q_3^2 & 2(q_2q_3-q_0q_1) \\ 2(q_3q_1-q_0q_2) & 2(q_3q_2+q_0q_1) & q_0^2-q_1^2-q_2^2+q_3^2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix},$$

or

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cos\Omega+l_1^2(1-\cos\Omega) & l_1l_2(1-\cos\Omega)-l_3\sin\Omega & l_1l_3(1-\cos\Omega)+l_2\sin\Omega \\ l_2l_1(1-\cos\Omega)+l_3\sin\Omega & \cos\Omega+l_2^2(1-\cos\Omega) & l_2l_3(1-\cos\Omega)-l_1\sin\Omega \\ l_3l_1(1-\cos\Omega)-l_2\sin\Omega & l_3l_2(1-\cos\Omega)+l_1\sin\Omega & \cos\Omega+l_3^2(1-\cos\Omega) \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

Grassmann's outer product algebra

Hermann Günther Grassmann (1809–1877),

Outer product and multivectors

- Consider an algebra generated by 1 and symbols e_1, e_2, and e_3 and require that the product (*outer* or *exterior* product) be subject to the rule

$$e_i \wedge e_j = -e_j \wedge e_i, \quad i,j = 1,2,3. \qquad \text{In particular, } e_i \wedge e_i = 0,\ i = 1,\ 2,\ 3.$$

  - The outer product is *anticommutative*.
- Each member (*multivector*) $\mathcal{C}$ of this algebra is a formal sum in the form

$$\mathcal{C} = \underbrace{\alpha}_{\text{scalarpart}} + \underbrace{a_1 e_1 + a_2 e_2 + a_3 e_3}_{\text{vectorpart}} + \underbrace{b_1 e_2 \wedge e_3 + b_2 e_3 \wedge e_1 + b_3 e_1 \wedge e_2}_{\text{bivectorpart}} + \underbrace{c e_1 \wedge e_2 \wedge e_3}_{\text{trivectorpart}}.$$

  - The symbol e_i is interpreted to be the (oriented) unit vector along the x_i-axis.
  - The *bivector* $e_i \wedge e_j$ is interpreted to be the (oriented) plane spanned by e_i and e_j.
  - The *trivector* $e_i \wedge e_j \wedge e_k$ is interpreted to be the (oriented) volume spanned by e_i, e_j, and e_k.
  - The Grassmann algebra is an 8-D vector space.

	1	e_1	e_2	e_3	$e_2 \wedge e_3$	$e_3 \wedge e_1$	$e_1 \wedge e_2$	$e_1 \wedge e_2 \wedge e_3$
1	1	e_1	e_2	e_3	$e_2 \wedge e_3$	$e_3 \wedge e_1$	$e_1 \wedge e_2$	$e_1 \wedge e_2 \wedge e_3$
e_1	e_1	0	$e_1 \wedge e_2$	$-e_3 \wedge e_1$	$e_1 \wedge e_2 \wedge e_3$	0	0	0
e_2	e_2	$-e_1 \wedge e_2$	0	$e_2 \wedge e_3$	0	0	0	0
e_3	e_3	$e_3 \wedge e_1$	$-e_2 \wedge e_3$	0	0	0	0	0
$e_2 \wedge e_3$	$e_2 \wedge e_3$	$e_1 \wedge e_2 \wedge e_3$	0	0	0	0	0	0
$e_3 \wedge e_1$	$e_3 \wedge e_1$	0	$e_3 \wedge e_1 \wedge e_2$	0	0	0	0	0
$e_1 \wedge e_2$	$e_1 \wedge e_2$	0	0	$e_1 \wedge e_2 \wedge e_3$	0	0	0	0
$e_1 \wedge e_2 \wedge e_3$	$e_1 \wedge e_2 \wedge e_3$	0	0	0	0	0	0	0

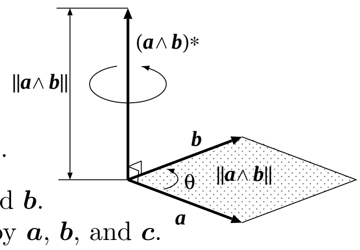
Vector calculus of Grassmann's algebra

- For vectors $\boldsymbol{a} = a_1e_1 + a_2e_2 + a_3e_3$, $\boldsymbol{b} = b_1e_1 + b_2e_2 + b_3e_3$, and $\boldsymbol{c} = c_1e_1 + c_2e_2 + c_3e_3$,

$$\boldsymbol{a} \wedge \boldsymbol{b} = (a_2b_3 - a_3b_2)e_2 \wedge e_3 + (a_3b_1 - a_1b_3)e_3 \wedge e_1 + (a_1b_2 - a_2b_1)e_1 \wedge e_2,$$

$$\boldsymbol{a} \wedge \boldsymbol{b} \wedge \boldsymbol{c} = (a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1 - a_2b_1c_3 - a_1b_3c_2)e_1 \wedge e_2 \wedge e_3.$$

- Their square norms are defined by

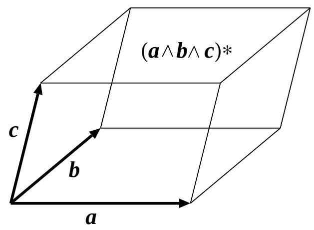


$$\|\boldsymbol{a} \wedge \boldsymbol{b}\|^2 = (a_2b_3 - a_3b_2)^2 + (a_3b_1 - a_1b_3)^2 + (a_1b_2 - a_2b_1)^3,$$

$$\|\boldsymbol{a} \wedge \boldsymbol{b} \wedge \boldsymbol{c}\|^2 = (a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1 - a_2b_1c_3 - a_1b_3c_2)^2.$$

  - $\|\boldsymbol{a} \wedge \boldsymbol{b}\|^2$ equals the squared area of the parallelogram defined by $\boldsymbol{a}$ and $\boldsymbol{b}$.
  - $\|\boldsymbol{a} \wedge \boldsymbol{b} \wedge \boldsymbol{c}\|^2$ equals the squared volume of the parallelepiped defined by $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$.

- If we define the *duals* of bivectors and trivectors by



$$(e_2 \wedge e_3)^* = e_1, \quad (e_3 \wedge e_1)^* = e_2, \quad (e_1 \wedge e_2)^* = e_3, \qquad (e_1 \wedge e_2 \wedge e_3)^* = 1,$$

then

$$(\boldsymbol{a} \wedge \boldsymbol{b})^* = \boldsymbol{a} \times \boldsymbol{b} \quad \text{(vector product)},$$

$$(\boldsymbol{a} \wedge \boldsymbol{b} \wedge \boldsymbol{c})^* = |\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}| \quad \text{(scalar triple product)}.$$

Direct and dual representations

- The *equation* of an object is an equality satisfied by the position vector $\boldsymbol{x}$ if and only if it belongs to that object.
- If it has the form $\boldsymbol{x} \wedge (\cdots) = 0$, the expression $(\cdots)$ is called the *direct representation*.
- If it has the form $\boldsymbol{x} \cdot (\cdots) = 0$, the expression $(\cdots)$ is called the *dual representation*.

subspace	direct representation	dual representation
origin	scalar α	trivector $\boldsymbol{a} \wedge \boldsymbol{b} \wedge \boldsymbol{c}$
line	vector $\boldsymbol{a}$	bivector $\boldsymbol{b} \wedge \boldsymbol{c}$
plane	bivector $\boldsymbol{a} \wedge \boldsymbol{b}$	vector $\boldsymbol{c}$
space	trivector $\boldsymbol{a} \wedge \boldsymbol{b} \wedge \boldsymbol{c}$	scalar α
equation	$\boldsymbol{x} \wedge (\cdots) = 0$	$\boldsymbol{x} \cdot (\cdots) = 0$

Clifford's geometric algebra

William Kingdon Clifford (1845–1879)

Geometric product and multivectors

- Consider an algebra generated by 1 and symbols e_1, e_2, and e_3 and require that the product (*geometric* or *Clifford* product) be subject to the rule

$$e_1^2 = e_2^2 = e_3^2 = 1, \qquad e_i e_j = -e_j e_i, \qquad i \neq j.$$

  - The geometric product is not commutative.
- Each member (*multivector*) $\mathcal{C}$ of this algebra is a formal sum in the form

$$\mathcal{C} = \underbrace{\alpha}_{\text{scalarpart}} + \underbrace{a_1 e_1 + a_2 e_2 + a_3 e_3}_{\text{vectorpart}} + \underbrace{b_1 e_2 e_3 + b_2 e_3 e_1 + b_3 e_1 e_2}_{\text{bivectorpart}} + \underbrace{c e_1 e_2 e_3}_{\text{trivectorpart}} .$$

  - The Clifford algebra is an 8-D vector space.

	1	e_1	e_2	e_3	e_2e_3	e_3e_1	e_1e_2	$e_1e_2e_3$
1	1	e_1	e_2	e_3	e_2e_3	e_3e_1	e_1e_2	$e_1e_2e_3$
e_1	e_1	1	e_1e_2	$-e_3e_1$	$e_1e_2e_3$	$-e_3$	e_2	e_2e_3
e_2	e_2	$-e_1e_2$	1	e_2e_3	e_3	$e_1e_2e_3$	$-e_2$	e_3e_1
e_3	e_3	e_3e_1	$-e_2e_3$	1	$-e_2$	e_1	$e_1e_2e_3$	e_1e_2
e_2e_3	e_2e_3	$e_1e_2e_3$	$-e_3$	e_2	-1	$-e_1e_2$	e_3e_1	$-e_1$
e_3e_1	e_3e_1	e_3	$e_3e_1e_2$	$-e_1$	e_1e_2	-1	$-e_2e_3$	$-e_2$
e_1e_2	e_1e_2	$-e_2$	e_1	$e_1e_2e_3$	$-e_3e_1$	e_2e_3	-1	$-e_3$
$e_1e_2e_3$	$e_1e_2e_3$	e_2e_3	e_3e_1	e_1e_2	$-e_1$	$-e_2$	$-e_3$	-1

Parity of the Clifford algebra

- A multivector consisting of an odd number of basis vectors is called an *odd multivector*:

$$\mathcal{A} = \underbrace{a_1 e_1 + a_2 e_2 + a_3 e_3}_{\text{vectorpart}} + \underbrace{c e_1 e_2 e_3}_{\text{trivectorpart}} .$$

  A multivector consisting of an even number of basis vectors is called an *even multivector*:

$$\mathcal{B} = \underbrace{\alpha}_{\text{scalar}} + \underbrace{b_1 e_2 e_3 + b_2 e_3 e_1 + b_3 e_1 e_2}_{\text{bivectorpart}} .$$

- The *parities* of multivectors are preserved by geometric products:
  - (odd multivector)(odd multivector) = (even multivector).
  - (even multivector)(even multivector) = (even multivector). ← Attention!
  - (odd multivector)(even multivector) = (odd multivector).

Hamilton's algebra ⊂ Clifford's algebra

- The geometric product of even multivectors is an even multivector.
  - The set of even multivectors is a *subalgebra* of Clifford's algebra.
- The subalgebra of even multivectors is the same as (i.e., *isomorphic* to) Hamilton's algebra:
  - If we let

$$i \equiv -e_2 e_3, \qquad j \equiv -e_3 e_1, \qquad k \equiv -e_1 e_2,$$

then

$$i^2 = j^2 = k^2 = -1,$$

$$ij = k, \qquad jk = i, \qquad ki = j,$$

$$ji = -ij, \qquad kj = -jk, \qquad ik = -ki.$$

	1	i	j	k
1	1	i	j	k
i	i	-1	k	$-j$
j	j	$-k$	-1	i
k	k	j	$-i$	-1

Grassmann's algebra $\subset$ Clifford's algebra

- For vectors $\boldsymbol{a} = a_1e_1 + a_2e_2 + a_3e_3$ and $\boldsymbol{b} = b_1e_1 + b_2e_2 + b_3e_3$,

$$\boldsymbol{ab} = a_1b_1 + a_2b_2 + a_3b_3 + (a_2b_3 - a_3b_2)e_2e_3 + (a_3b_1 - a_1b_3)e_3e_1 + (a_1b_2 - a_2b_1)e_1e_2,$$

$$\boldsymbol{ba} = b_1a_1 + b_2a_2 + b_3a_3 + (b_2a_3 - b_3a_2)e_2e_3 + (b_3a_1 - b_1a_3)e_3e_1 + (b_1a_2 - b_2a_3)e_1e_2.$$

- We define the outer product of vectors by *antisymmetrization*:

$$\boldsymbol{a} \wedge \boldsymbol{b} = \frac{1}{2}(\boldsymbol{ab} - \boldsymbol{ba}), \qquad \boldsymbol{a} \wedge \boldsymbol{b} \wedge \boldsymbol{c} = \frac{1}{6}(\boldsymbol{abc} + \boldsymbol{bca} + \boldsymbol{cab} - \boldsymbol{cba} - \boldsymbol{bac} - \boldsymbol{acb}).$$

  - The outer product of four or more vectors is defined to be 0: $\boldsymbol{a} \wedge \boldsymbol{b} \wedge \boldsymbol{c} \wedge \boldsymbol{d} = \cdots = 0$.
  - All the rules of the outer product are satisfied.
- For vectors $\boldsymbol{a} = a_1e_1 + a_2e_2 + a_3e_3$, $\boldsymbol{b} = b_1e_1 + b_2e_2 + b_3e_3$, and $\boldsymbol{c} = c_1e_1 + c_2e_2 + c_3e_3$,

$$\boldsymbol{a} \wedge \boldsymbol{b} = (a_2b_3 - a_3b_2)e_2e_3 + (a_3b_1 - a_1b_3)e_3e_1 + (a_1b_2 - a_2b_1)e_1e_2,$$

$$\boldsymbol{a} \wedge \boldsymbol{b} \wedge \boldsymbol{c} = (a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1 - a_2b_1c_3 - a_1b_3c_2)e_1e_2e_3.$$

- It turns out that *symmetrization* of the vector products is the *inner product*:

$$\frac{1}{2}(\boldsymbol{ab} + \boldsymbol{ba}) = a_1b_1 + a_2b_2 + a_3b_3 = \langle \boldsymbol{a}, \boldsymbol{b} \rangle. \qquad \text{In particular, } \boldsymbol{a}^2 = \|\boldsymbol{a}\|^2.$$

- Hence, the geometric product $\boldsymbol{ab}$ has the expression

$$\boldsymbol{ab} = \langle \boldsymbol{a}, \boldsymbol{b} \rangle + \boldsymbol{a} \wedge \boldsymbol{b}.$$

  - *The geometric product computes the inner product $\langle \boldsymbol{a}, \boldsymbol{b} \rangle$ and the outer product $\boldsymbol{a} \wedge \boldsymbol{b}$ at the same time.*

Inverses

- Since $\boldsymbol{a}^2 = \|\boldsymbol{a}\|^2$, we have

$$\boldsymbol{a}\frac{\boldsymbol{a}}{\|\boldsymbol{a}\|^2} = \frac{\boldsymbol{a}}{\|\boldsymbol{a}\|^2}\boldsymbol{a} = 1$$

  Hence, every vector $\boldsymbol{a}$ $(\neq 0)$ has its *inverse*:

$$\boldsymbol{a}^{-1} = \frac{\boldsymbol{a}}{\|\boldsymbol{a}\|^2}, \qquad \boldsymbol{a}\boldsymbol{a}^{-1} = \boldsymbol{a}^{-1}\boldsymbol{a} = 1.$$

- The inverse of a product is the product of the inverses in the reversed order:

$$(\boldsymbol{a}\boldsymbol{b}\boldsymbol{c}\cdots)^{-1} = \cdots\boldsymbol{c}^{-1}\boldsymbol{b}^{-1}\boldsymbol{a}^{-1}$$

- Bivector $\boldsymbol{a}\wedge\boldsymbol{b}$ and trivector $\boldsymbol{a}\wedge\boldsymbol{b}\wedge\boldsymbol{c}$ have the following inverses:

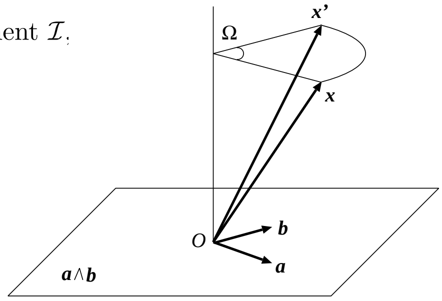
$$(\boldsymbol{a}\wedge\boldsymbol{b})^{-1} = \frac{\boldsymbol{b}\wedge\boldsymbol{a}}{\|\boldsymbol{a}\wedge\boldsymbol{b}\|^2}, \qquad (\boldsymbol{a}\wedge\boldsymbol{b}\wedge\boldsymbol{c})^{-1} = \frac{\boldsymbol{c}\wedge\boldsymbol{b}\wedge\boldsymbol{a}}{\|\boldsymbol{a}\wedge\boldsymbol{b}\wedge\boldsymbol{c}\|^2}.$$

Rotation

- Rotation is specified by an (oriented) plane in which the rotation takes place and the angle Ω, whose sense follows the orientation of that plane.
- An oriented plane is defined by a bivector $\boldsymbol{a} \wedge \boldsymbol{b}$. The sense is so defined that $\boldsymbol{a}$ approaches toward $\boldsymbol{b}$. We define the (oriented) *surface element* $\mathcal{I}$ by

$$\mathcal{I} = \frac{\boldsymbol{a} \wedge \boldsymbol{b}}{\|\boldsymbol{a} \wedge \boldsymbol{b}\|}.$$

  - The surface element $\mathcal{I}$ does *not* depend on the choice of $\boldsymbol{a}$ and $\boldsymbol{b}$ *as long as they define the same plane with the same sense.*
- If a vector $\boldsymbol{x}$ is rotated to $\boldsymbol{x}'$ by angle Ω in the plane of surface element $\mathcal{I}$,



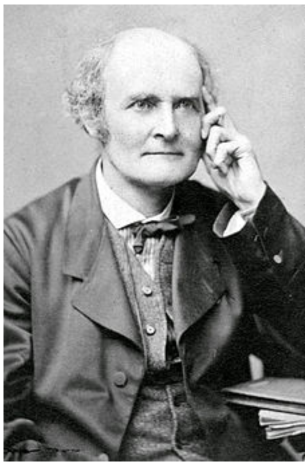
$$\boldsymbol{x}' = \mathcal{R}\boldsymbol{x}\mathcal{R}^{-1}, \qquad \mathcal{R} \equiv \cos\frac{\Omega}{2} - \mathcal{I}\sin\frac{\Omega}{2}.$$

  - It can be shown that $\mathcal{I}^2 = -1$ and the inverse of $\mathcal{R}$ is given by

$$\mathcal{R}^{-1} = \cos\frac{\Omega}{2} + \mathcal{I}\sin\frac{\Omega}{2}.$$

Grassmann–Cayley algebra

Hermann Günther Grassmann (1809–1877)

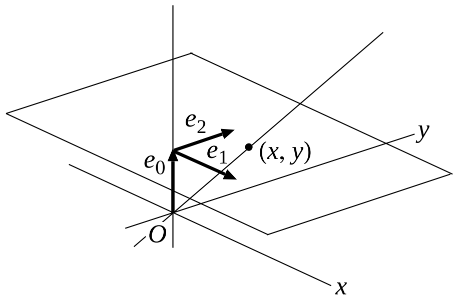


Arthur Cayley (1821–1895)

4-D homogeneous space

- Consider an algebra generated by 1 and symbols e_0, e_1, e_2, and e_3 with the outer product subject to the rule

$$e_i \wedge e_j = -e_j \wedge e_i, \qquad i, j = 0, 1, 2, 3.$$



- A point (x, y, z) in 3-D space is represented by

$$p = e_0 + xe_1 + ye_2 + ze_3.$$

  - The symbol e_0 is identified with the *origin* of the 3-D space, since (0,0,0) is represented by e_0.
- The expression

$$\boldsymbol{u} = u_1e_1 + u_2e_2 + u_3e_3$$

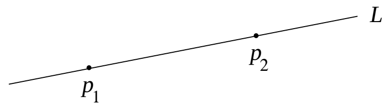
  indicates the direction in 3-D and is interpreted to be the *point at infinity* in that direction.
  - The magnitude is irrelevant: $\boldsymbol{u}$ and $\alpha\boldsymbol{u}$ for any $\alpha \neq 0$ are regarded as the same direction and the same point at infinity.

Lines

- The line L passing through 3-D positions $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ are represented by the bivector

  $$L = p_1 \wedge p_2,$$

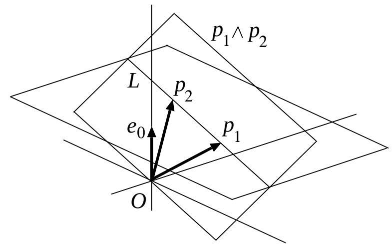
  where p_i is the representations of $\boldsymbol{x}_i$; $p_i = e_0 + \boldsymbol{x}_i$.



- The line L with orientation $\boldsymbol{u}$ passing through 3-D position $\boldsymbol{x}$ is represented by the bivector

  $$L = \boldsymbol{u} \wedge p.$$

  – Bivectors $p_1 \wedge p_2$ and $\boldsymbol{u} \wedge p$ both define the same 2-D subspace in 4-D.



Expression of lines

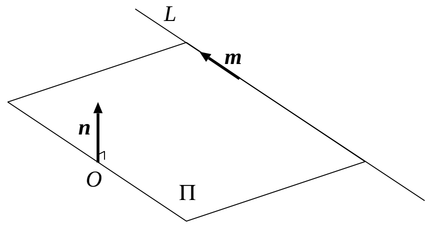
After expansion, the bivector L that represents a line has the following expression:

$$L = m_1 e_0 \wedge e_1 + m_2 e_0 \wedge e_2 + m_3 e_0 \wedge e_3 + n_1 e_2 \wedge e_3 + n_2 e_3 \wedge e_1 + n_3 e_1 \wedge e_2.$$

- m_i, n_i, $i = 1, 2, 3$, are called the *Plücker coordinates* of L.
- The vector $\boldsymbol{m} = m_1 e_1 + m_2 e_2 + m_3 e_3$ indicates the *direction* of the line L.
- The vector $\boldsymbol{n} = n_1 e_1 + n_2 e_2 + n_3 e_3$ is the *surface normal* to the *supporting plane* of L ($\equiv$ the plane passing through L and the origin O).
- In terms of $\boldsymbol{m}$ and $\boldsymbol{n}$, the bivector L can be written as

$$L = e_0 \wedge \boldsymbol{m} - \boldsymbol{n}^*$$

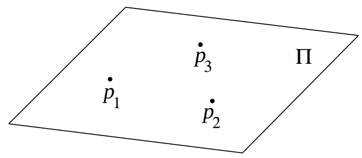
  - For $\boldsymbol{n} = n_1 e_1 + n_2 e_2 + n_3 e_3$, its dual is $\boldsymbol{n}^* = -n_1 e_2 \wedge e_3 - n_2 e_3 \wedge e_1 - n_3 e_1 \wedge e_2$.



Planes

- The plane Π passing through 3-D positions $\boldsymbol{x}_1$, $\boldsymbol{x}_2$, and $\boldsymbol{x}_3$ is represented by the trivector

$$\Pi = p_1 \wedge p_2 \wedge p_3.$$



- The plane Π passing through 3-D positions $\boldsymbol{x}_1$, $\boldsymbol{x}_2$, and $\boldsymbol{x}_3$ and containing orientation $\boldsymbol{u}$ is represented by the trivector

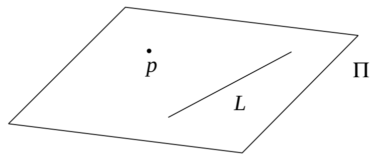
$$\Pi = \boldsymbol{u} \wedge p_1 \wedge p_2.$$

- The plane Π passing through 3-D position $\boldsymbol{x}$ and containing orientation $\boldsymbol{u}$ and $\boldsymbol{v}$ is represented by the trivector

$$\Pi = \boldsymbol{u} \wedge \boldsymbol{v} \wedge p.$$

- The plane Π passing through line L and 3-D position $\boldsymbol{x}$ is represented by the trivector

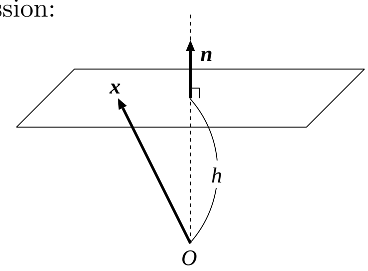
$$\Pi = L \wedge p.$$



  - Trivectors $p_1 \wedge p_2 \wedge p_3$, $\boldsymbol{u} \wedge p_1 \wedge p_2$, $\boldsymbol{u} \wedge \boldsymbol{v} \wedge p$, and $L \wedge p$ all define the same 3-D subspace in 4-D.

Expressions of planes

After expansion, the trivector Π that represents a plane has the following expression:



$$\Pi = n_1 e_0 \wedge e_2 \wedge e_3 + n_2 e_0 \wedge e_3 \wedge e_1 + n_3 e_0 \wedge e_1 \wedge e_2 + h e_1 \wedge e_2 \wedge e_3$$

- n_i, $i = 1, 2, 3$, and h are called the *Plücker coordinates* of Π.
- The vector $\boldsymbol{n} = n_1 e_1 + n_2 e_2 + n_3 e_3$ is the *surface normal* to the plane Π.
- The value h is the *distance* of the plane Π from the origin O.
- In terms of $\boldsymbol{n}$ and h, the trivector Π can be written as

$$\Pi = -e_0 \wedge \boldsymbol{n}^* + hI, \qquad I \equiv e_1 \wedge e_2 \wedge e_3 \ \ (\textit{volume element}).$$

$$\boldsymbol{n}^* = -n_1 e_2 \wedge e_3 - n_2 e_3 \wedge e_1 - n_3 e_1 \wedge e_2.$$

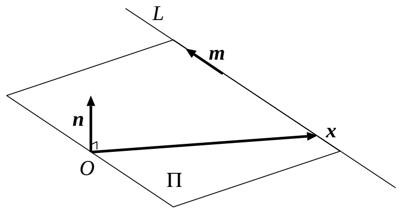
Equations of lines and planes

- A point p is on line L if and only if

  $$p \wedge L = 0 \ \text{ (“equation” of the line } L\text{).}$$

  - In terms of the Plücker coordinates, this is equivalently written as

    $$\boldsymbol{x} \times \boldsymbol{m} = \boldsymbol{n}.$$

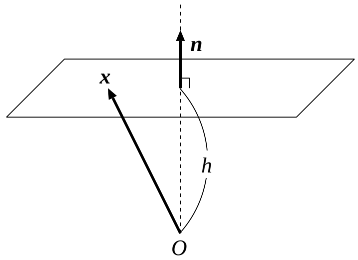


- A point p is on plane Π if and only if

  $$p \wedge \Pi = 0 \ \text{ (“equation” of the plane } \Pi\text{).}$$

  - In terms of the Plücker coordinates, this is equivalently written as

    $$\langle \boldsymbol{n}, \boldsymbol{x} \rangle = h.$$



Direct and dual representations

		direct	dual
points	representation	$q = e_0 + \boldsymbol{y}$	$q^* = e_0 \wedge \boldsymbol{y}^* + I$
	equation	$p \wedge q = 0$	$p \cdot q^* = 0$
lines	representation	$L = e_0 \wedge \boldsymbol{m} - \boldsymbol{n}^*$	$L^* = -e_0 \wedge \boldsymbol{n} + \boldsymbol{m}^*$
	equation	$p \wedge L = 0$	$p \cdot L^* = 0$
planes	representation	$\Pi = -e_0 \wedge \boldsymbol{n}^* + hI$	$\Pi^* = he_0 - \boldsymbol{n}$
	equation	$p \wedge \Pi = 0$	$p \cdot \Pi^* = 0$

Duals

- The dual of point $p = e_0 + \boldsymbol{x}$ is a plane represented by trivector

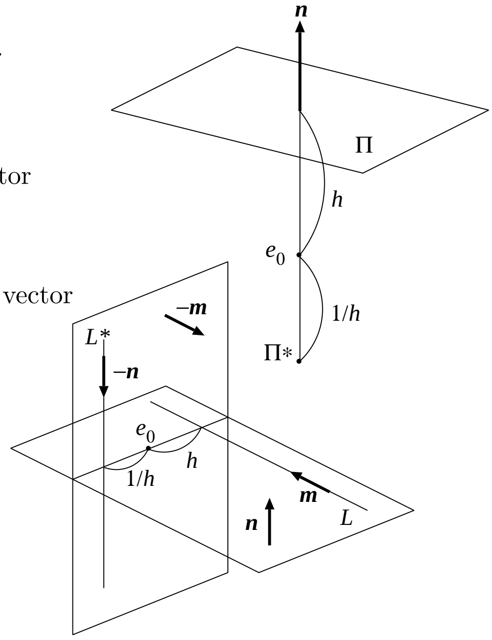
  $$p^* = e_0 \wedge \boldsymbol{x}^* + I \ \ (\textit{dual plane}).$$

- The dual of line $L = e_0 \wedge \boldsymbol{m} - \boldsymbol{n}^*$ is a line represented by bivector

  $$L^* = -e_0 \wedge \boldsymbol{n} + \boldsymbol{m}^* \ \ (\textit{dual line}).$$

- The dual of plane $\Pi = -e_0 \wedge \boldsymbol{n}^* + hI$ is a point represented by vector

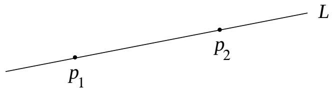
  $$\Pi^* = he_0 - \boldsymbol{n} \ \ (\textit{dual point}).$$



Join

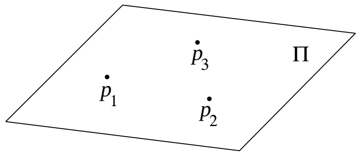
- The *join* of points p_1 and p_2 is the line L passing through them:

$$L = p_1 \cup p_2 \;\; (= p_1 \wedge p_2).$$



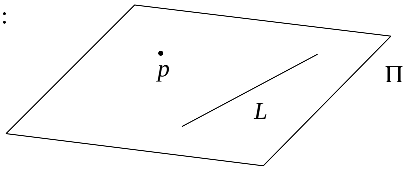
- The *join* of points p_1, p_2, and p_3 is the plane Π pas passing through them:

$$\Pi = p_1 \cup p_2 \cup p_3 \;\; (= p_1 \wedge p_2 \wedge p_3).$$



- The *join* of a point p and a line L is the plane Π passing through them:

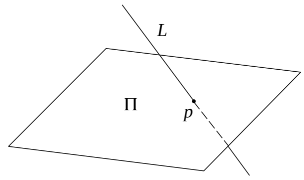
$$\Pi = p \cup L \;\; (= p \wedge L).$$



Meet

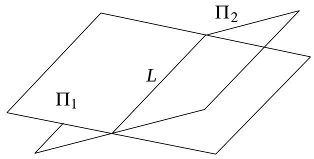
- The *meet* of a line L and a plane Π is their intersection point p:

$$p = L \cap \Pi.$$



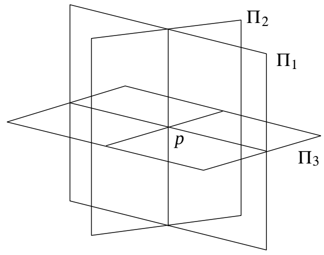
- The *meet* of planes Π_1 and Π_2 is their intersection line L:

$$L = \Pi_1 \cap \Pi_2.$$



- The *meet* of planes Π_1, Π_2, and Π_3 is their intersection point p:

$$p = \Pi_1 \cap \Pi_2 \cap P_3.$$



Duality theorem

- The dual of the join of points p_1 and p_2 is the meet of their dual planes p_1^* and p_2^*:

  $$(p_1 \cup p_2)^* = p_1^* \cap p_2^*.$$

  Conversely, the dual of the meet of planes Π_1 and Π_2 is the join of their dual points Π_1^* and Π_2^*:

  $$(\Pi_1 \cap \Pi_2)^* = \Pi_1^* \cup \Pi_2^*.$$

- The dual of the join of a point p_1 and a line L is the meet of the dual plane p^* and the dual line L^*:

  $$(p \cup L)^* = p^* \cap L^*.$$

  Conversely, the dual of the meet of a line L and a plane Π is the join of their dual line L^* and the dual point Π^*:

  $$(L \cap \Pi)^* = L^* \cup \Pi^*.$$

Summary of duality

	point p_2	line L_2	plane Π_2
point p_1	$p_1 \cup p_2 = p_1 \wedge p_2$	$p_1 \cup L_2 = p_1 \wedge L_2$	—
line L_1	$L_1 \cup p_2 = L_1 \wedge p_2$	—	$L_1 \cap \Pi_2 = (L_1^* \wedge \Pi_2^*)^*$
plane Π_1	—	$\Pi_1 \cap L_2 = (\Pi_1^* \wedge L_2^*)^*$	$\Pi_1 \cap \Pi_2 = (\Pi_1^* \wedge \Pi_2^*)^*$

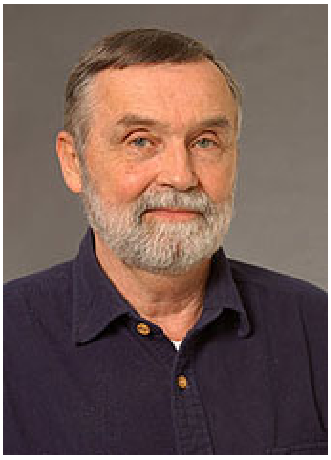
- The dual of the join of points p_1, p_2, and p_3 is the meet of their dual planes Π_1, Π_2, and Π_3:

  $$(p_1 \cup p_2 \cup p_3)^* = p_1^* \cap p_2^* \cap p_3^*.$$

  Conversely, the dual of the meet of planes Π_1, Π_2, and Π_3 is the join of their dual points Π_1^*, Π_2^*, and Π_3^*:

  $$(\Pi_1 \cap \Pi_2 \cap \Pi_3)^* = \Pi_1^* \cup \Pi_2^* \cup \Pi_3^*.$$

Conformal geometric algebra



David Hestenes (1933–)

5-D non-Euclidean conformal space

- Consider an algebra generated by 1 and symbols e_0, e_1, e_2, e_3, and e_∞ with the geometric product subject to the rule:

$$e_1^2 = e_2^2 = e_3^2 = 1, \qquad e_0^2 = e_\infty^2 = 0, \qquad e_0 e_\infty + e_\infty e_0 = -2,$$

$$e_i e_0 + e_0 e_i = e_i e_\infty + e_\infty e_i = 0, \qquad e_i e_j + e_j e_i = 0, \qquad i, j = 1, 2, 3$$

- We identify a 3-D vector $\boldsymbol{x}$ with $\boldsymbol{x} = x_1 e_1 + x_2 e_2 + x_3 e_3$ and call an element in the form

$$x = x_0 e_0 + \boldsymbol{x} + x_\infty e_\infty$$

  simply a (5-D)"vector".
- The inner product of vectors is defined by *symmetrization* of the geometric product:

$$\langle x, y \rangle \equiv \frac{1}{2}(xy + yx) = \langle \boldsymbol{x}, \boldsymbol{y} \rangle - x_0 y_\infty - x_\infty y_0$$

- The square norm is defined by

$$\|x\|^2 = \langle x, x \rangle = x^2 = \|\boldsymbol{x}\|^2 - 2x_0 x_\infty. \qquad \text{This can be negative (\textit{Minkowski norm}).}$$

  - A norm that is positive for all nonzero vectors is called a *Euclidean metric*. A space with a Euclidean metric is said to be a *Euclidean space*.
  - Conformal geometry is realized in an non-Euclidean space.

Outer product

- The outer product of vectors is defined by *antisymmetrization* of the geometric product:

$$
\begin{aligned}
x \wedge y &\equiv \frac{1}{2}(xy - yx), \\
x \wedge y \wedge z &\equiv \frac{1}{6}(xyz + yzx + zxy - zyx - yxz - xzy), \\
x \wedge y \wedge z \wedge w &\equiv \frac{1}{24}(xyzw - yxzw + yzxw - yzwx + \cdots), \\
x \wedge y \wedge z \wedge w \wedge u &\equiv \frac{1}{120}(xyzwu - yxzwu + yzxwu - \cdots).
\end{aligned}
$$

  - All the properties of the Grassmann outer product are satisfied.
  - The outer product of six or more elements is zero.
- The geometric product of vectors is expressed as the sum of the inner product (symmetric part) and the outer product (antisymmetric part):

$$xy = \langle x, y \rangle + x \wedge y.$$

- *Orthogonal* vectors x and y (i.e., $\langle x, y \rangle = 0$) are *anticommutative*: $xy = -yx$.

Representation of geometric objects

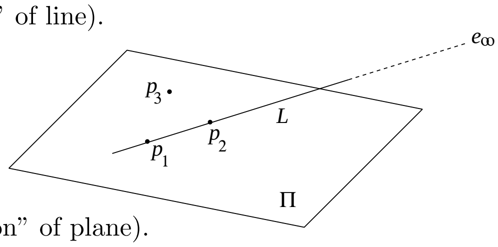
- A 3-D point $\boldsymbol{x}$ is represented by a vector in the form

$$p = e_0 + \boldsymbol{x} + \frac{1}{2}\|\boldsymbol{x}\|^2 e_\infty.$$

  - All points are represented by *null vectors*: $\|p\|^2 = 0$.
  - The inner product of points is their negative half square distance: $\langle p, q\rangle = -\dfrac{1}{2}\|\boldsymbol{x} - \boldsymbol{y}\|^2$.
- The line L passing through two points p_1 and p_2 is represented by the trivector

$$L = p_1 \wedge p_2 \wedge e_\infty.$$

  - A point p is on line L if and only if $p \wedge L = 0$ ("equation" of line).
  - Any line L passes through the infinity: $e_\infty \wedge L = 0$.



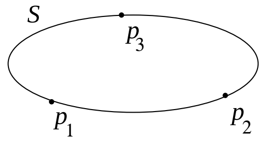
- The plane Π passing through three points p_1, p_2, and p_3 is represented by the 4-vector

$$\Pi = p_1 \wedge p_2 \wedge p_3 \wedge e_\infty.$$

  - A point p is on plane Π if and only if $p \wedge \Pi = 0$ ("equation" of plane).
  - Any plane Π passes through the infinity: $e_\infty \wedge \Pi = 0$.

Circles and spheres

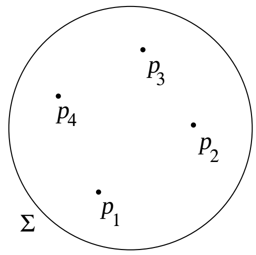
- A circle S passing through points p_1, p_2, and p_3 is represented by a trivector



$$S = p_1 \wedge p_2 \wedge p_3.$$

  - A point p is on circle S if and only if $p \wedge S = 0$ ("equation" of circle).
  - A line $L = p_1 \wedge p_2 \wedge e_\infty$ is interpreted to be the circle passing through p_1 and p_2 and the infinity e_∞.

- A sphere Σ passing through points p_1, p_2, p_3, and p_3 is represented by a 4-vector



$$\Sigma = p_1 \wedge p_2 \wedge p_3 \wedge p_4.$$

  - A point p is on sphere Σ if and only if $p \wedge \Sigma = 0$ ("equation" of sphere).
  - A plane $\Pi = p_1 \wedge p_2 \wedge p_3 \wedge e_\infty$ is interpreted to be the sphere passing through p_1, p_2, and p_3 and the infinity e_∞.

Direct and dual representations

object	direct representation	dual representation
(isolated) point	$p = e_0 + \boldsymbol{x} + \|\boldsymbol{x}\|^2 e_\infty/2$	$p = e_0 + \boldsymbol{x} + \|\boldsymbol{x}\|^2 e_\infty/2$
line	$p_1 \wedge p_2 \wedge e_\infty$	$\pi_1 \wedge \pi_2$
	$p \wedge \boldsymbol{u} \wedge e_\infty$	
plane	$p_1 \wedge p_2 \wedge p_3 \wedge e_\infty$	$\boldsymbol{n} + he_\infty$
	$p_1 \wedge p_2 \wedge \boldsymbol{u} \wedge e_\infty$	$p_1 - p_2$
	$p \wedge \boldsymbol{u}_1 \wedge \boldsymbol{u}_2 \wedge e_\infty$	
sphere	$p_1 \wedge p_2 \wedge p_3 \wedge p_4$	$c - r^2 e_\infty/2$
circle	$p_1 \wedge p_2 \wedge p_3$	$\sigma_1 \wedge \sigma_2$
		$\sigma \wedge \pi$
point pair	$p_1 \wedge p_2$	$s \wedge \sigma$
		$s \wedge \pi$
(flat) point	$p \wedge e_\infty$	$\pi \wedge l$
		$\pi_1 \wedge \pi_2 \wedge \pi_3$
equation	$p \wedge (\cdots) = 0$	$p \cdot (\cdots) = 0$

The outer product $\wedge$ indicates:

- *join for direct representations*,
- *meet for dual representations*.

(direct representation) $\wedge$ (direct representation) = (direct representation of their join),

(dual representation) $\wedge$ (dual representation) = (dual representation of their meet).

Versors

- The geometric product of vectors

  $$\mathcal{V} = v_k v_{k-1} \cdots v_1$$

  is called a *versor* when it acts on a geometric object in the form

  $$\mathcal{V}(\cdots)\mathcal{V}^\dagger,$$

  where the *conjugate* $\mathcal{V}^\dagger$ is defined by

  $$\mathcal{V}^\dagger \equiv (-1)^k \mathcal{V}^{-1} = (-1)^k v_1^{-1} v_2^{-1} \cdots v_k^{-1}. \qquad k\text{: the } \textit{grade} \text{ of the versor}$$

- Versors preserve the outer product up to sign:

  $$\mathcal{V}(x \wedge y \wedge \cdots \wedge z)\mathcal{V}^\dagger = (-1)^k (\mathcal{V}x\mathcal{V}^\dagger \wedge \mathcal{V}y\mathcal{V}^\dagger \wedge \cdots \wedge \mathcal{V}z\mathcal{V}^\dagger).$$

  - A sphere (incl. plane) is mapped to a sphere (incl. plane):

    $$p \wedge (p_1 \wedge p_2 \wedge p_3 \wedge p_4) = 0 \;\Rightarrow\; p' \wedge (p_1' \wedge p_2' \wedge p_3' \wedge p_4') = 0, \qquad p' = \mathcal{V}p\mathcal{V}^\dagger,\; p_i' = \mathcal{V}p_i\mathcal{V}^\dagger.$$

  - A circle (incl. line) is mapped to a circle (incl. line):

    $$p \wedge (p_1 \wedge p_2 \wedge p_3) = 0 \;\Rightarrow\; p' \wedge (p_1' \wedge p_2' \wedge p_3') = 0, \qquad p' = \mathcal{V}p\mathcal{V}^\dagger,\; p_i' = \mathcal{V}p_i\mathcal{V}^\dagger.$$

- The versors preserve the inner product:

  $$\langle x, y \rangle = \langle \mathcal{V}x\mathcal{V}^\dagger, \mathcal{V}y\mathcal{V}^\dagger \rangle,$$

Conformal mapping

- Versors induce *conformal mappings* (= angle-preserving mappings).
  - Spheres (incl. planes) are mapped to spheres (incl. planes).
  - Circles (incl. lines) are mapped to circles (incl. lines).
- Well known conformal mappings include:
  - Similarity
  - Rigid motion
  - Rotation
  - Reflection
  - Dilation
  - Translation
  - Identity
- A typical unconventional conformal mapping is (*spherical*) *inversion*.
- A conformal mapping is an *isometry* (= length preserving mapping) if and only if e_∞ is mapped to itself:

$$\mathcal{V} e_\infty \mathcal{V}^\dagger = e_\infty.$$

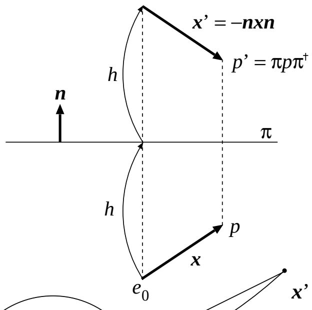
  - Well known isometries include rigid motion, rotation, reflection, translation, and identity

Examples of conformal versors

Reflector (grade 1)**:**

Reflection with respect to the plane of surface unit normal $\boldsymbol{n}$ and distance h from the origin.

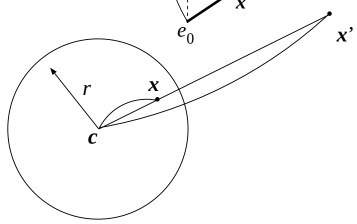
$$\pi = \boldsymbol{n} + he_\infty \ \ (= \pi^{-1}).$$



Invertor (grade 1)**:**

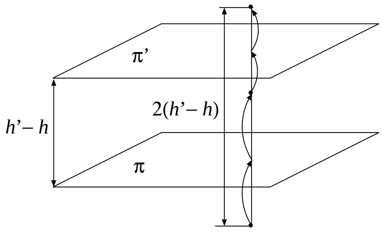
Inversion with respect to the sphere of center c and radius r.

$$\sigma = c - \frac{1}{2}r^2 e_\infty, \qquad \sigma^{-1} = \frac{\sigma}{r^2}.$$



Examples of conformal versors

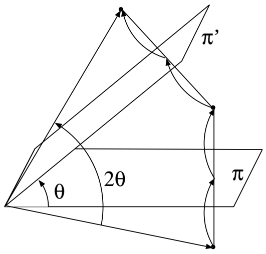
Translator (grade 2)**:** Translation by $\boldsymbol{t}$.



$$\mathcal{T}_{\boldsymbol{t}} = 1 - \frac{1}{2}\boldsymbol{t}e_\infty, \qquad \mathcal{T}_{\boldsymbol{t}}^{-1} = \mathcal{T}_{-\boldsymbol{t}} = 1 + \frac{1}{2}\boldsymbol{t}e_\infty.$$

- Translation is realized by consecutive reflections with respect to *parallel planes*.

Rotor (grade 2) Rotation in the plane with surface element $\mathcal{I}$ by angle Ω.

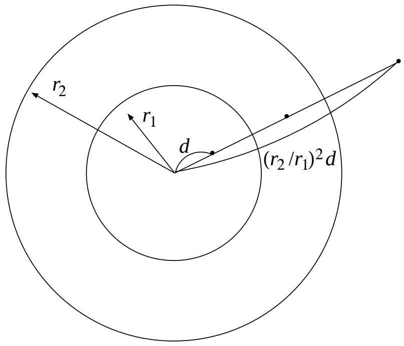


$$\mathcal{R} = \cos\frac{\Omega}{2} - \mathcal{I}\sin\frac{\Omega}{2}, \qquad \mathcal{R}^{-1} = \cos\frac{\Omega}{2} + \mathcal{I}\sin\frac{\Omega}{2}.$$

- Rotation is realized by consecutive reflections with respect to *intersecting planes*; the intersection is the axis of rotation.
- "Translation" is a rotation around an axis infinitely far away.

Dilator (grade 2)**:** Dilation around the origin by $e^{\gamma/2}$.

$$\mathcal{D} = \cosh\frac{\gamma}{2} + \mathcal{O}\sin\frac{\gamma}{2}, \qquad \mathcal{D}^{-1} = \cosh\frac{\gamma}{2} - \mathcal{O}\sin\frac{\gamma}{2}, \qquad \mathcal{O} \equiv e_0 \wedge e_\infty.$$



- Dilation is realized as consecutive inversions with respect to *concentric spheres*.

Motor (grade 4)**:** Rotation $\mathcal{R}$ followed by translation $\mathcal{T}_{\boldsymbol{t}}$.

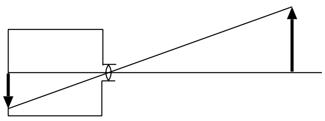
$$\mathcal{M} = \mathcal{T}_{\boldsymbol{t}}\mathcal{R}, \qquad \mathcal{M}^{-1} = \mathcal{R}^{-1}\mathcal{T}_{\boldsymbol{t}}^{-1}.$$

Versors in the conformal space

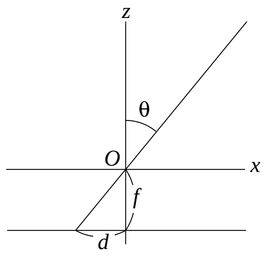
name	grade	expression
reflector	1	$\pi = \boldsymbol{n} + h e_\infty$
invertor	1	$\sigma = c - r^2 e_\infty / 2$
translator	2	$\mathcal{T}_{\boldsymbol{t}} = 1 - \boldsymbol{t} e_\infty / 2 = \exp(-\boldsymbol{t} e_\infty / 2)$
		consecutive reflections for parallel planes
rotor	2	$\mathcal{R} = \cos \Omega/2 - \mathcal{I} \sin \Omega/2 = \exp(-\mathcal{I}\Omega/2)$
		consecutive reflections for intersecting planes
dilator	2	$\mathcal{D} = \cosh \gamma/2 + O \sin \gamma/2 = \exp O\gamma/2$
		consecutive inversions for concentric spheres
motor	4	$\mathcal{M} = \mathcal{T}_{\boldsymbol{t}} \mathcal{R}$
		composition of rotation and translation

Camera imaging geometry

Perspective cameras



Camera imaging geometry



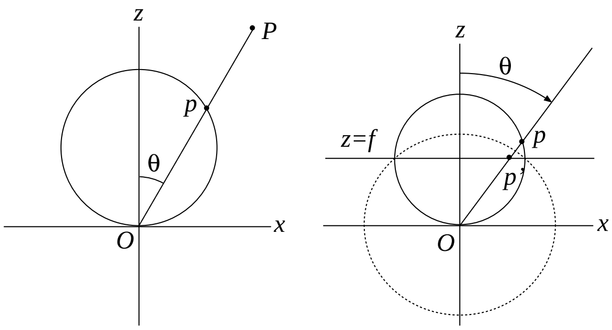
Perspective projection

$$d = f \tan\theta.$$

d :	distance from the principal point
f :	focal length
θ	incidence angle

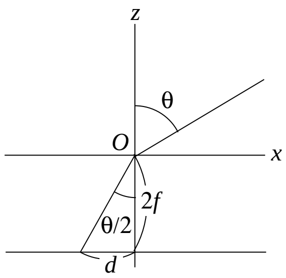
- A camera is a device to record the incoming *rays* of light.
  - The depth information is lost.
- The image need not be planar as long as rays are recorded.
  - It can be a sphere

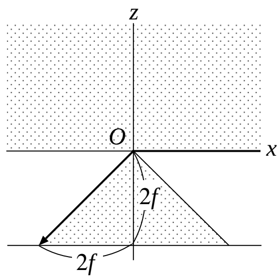
Image sphere and stereoscopic projection



- All rays coming in front of the camera are recorded on a sphere centered on the optical axis *passing through the lens center O*.
- The correspondence between the spherical and the planar images is a *stereographic projection* from the lens center O.
- It is also an *inversion* with respect to a sphere at O with radius $\sqrt{2}f$.

Fisheye lens cameras



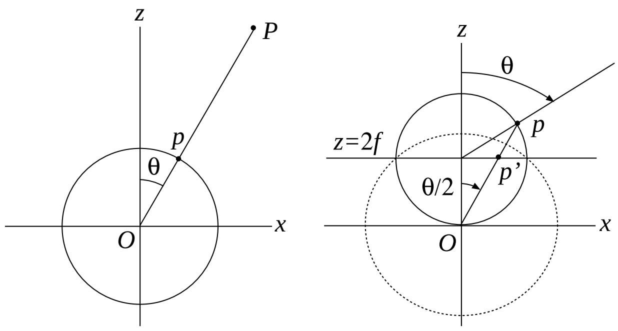


$$d = 2f \tan\frac{\theta}{2}.$$

d :	distance from the principal point
f :	focal length
θ	incidence angle

- All the scene in front is imaged within a circle of radius $2f$ around the principal point.
  - The outside is the image of the scene behind.
- In the neighborhood of the principal point $\theta \approx 0$, the projection is approximately perspective: $d \approx f \tan\theta$.

Image sphere and stereoscopic projection

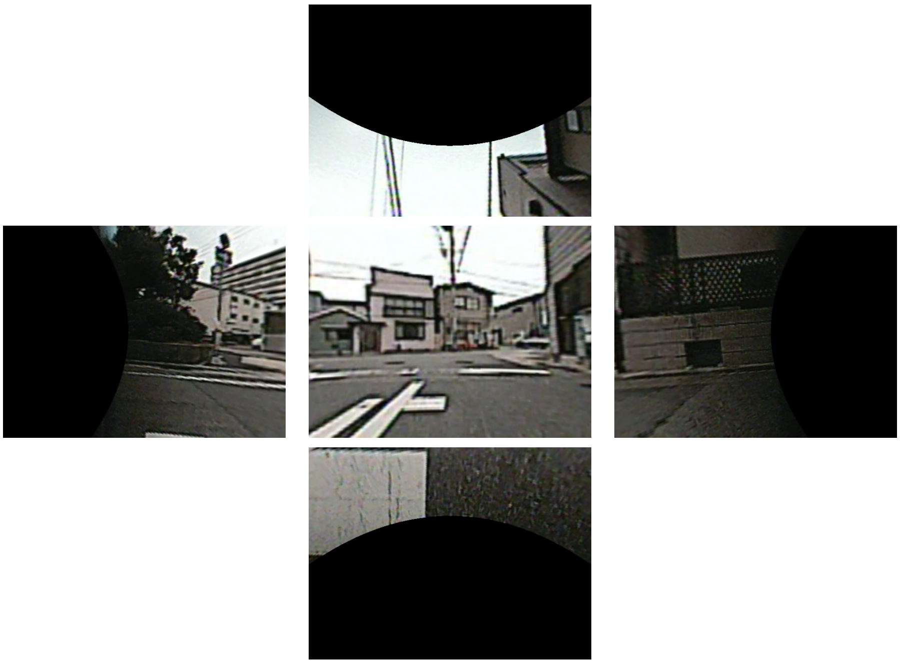


- All rays coming from around the camera are recorded on a sphere centered on the lens center O.
- The correspondence between the spherical and the planar images is a *stereographic projection* from the "south pole" O of the image sphere.
  - Cf. the relationship between the central and inscribed angles.
- It is also an *inversion* with respect to a sphere at O with radius $2\sqrt{2}f$.

Fisheye lens image example

A fisheye lens image of an outdoor scene.

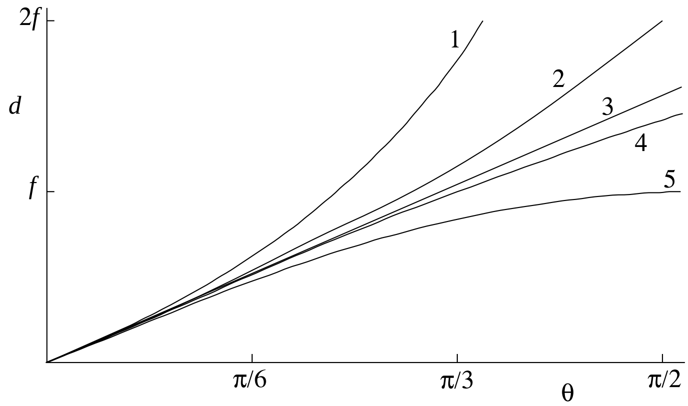
Transformation from fisheye to perspective



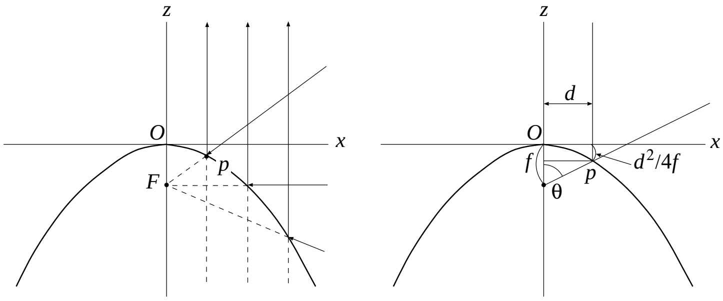
- We can transform a fisheye image to a perspective image as if obtained by rotating the camera by any angle.
- This technique can be used in various ways for vehicle-mounted camera applications.
  - Drivers are assisted by a fisheye lens camera mounted in front, warned of approaching vehicles right or left.
  - Using multiple fisheye lens cameras, we can generate an image of the ground surface around the vehicle as if seen from high above.

Other types of fisheye lens camera

	name	projection equation
1.	perspective projection	$d = f \tan\theta$
2.	stereographic projection	$d = 2f \tan\theta/2$
3.	ortogonal projection	$d = f \sin\theta$
4.	equisolid angle projection	$d = 2f \sin\theta/2$
5.	equidistance projection	$d = f\theta$



Omnidirectional cameras using a parabolic mirror



- Incoming rays toward the focus F are reflected as parallel rays upward.
  - Parallel rays coming upward from below would be reflected to converge at the focus F.

From the figure,

$$\tan\theta = \frac{d}{f - d^2/4f}.$$

From the tangent double-angle formula,

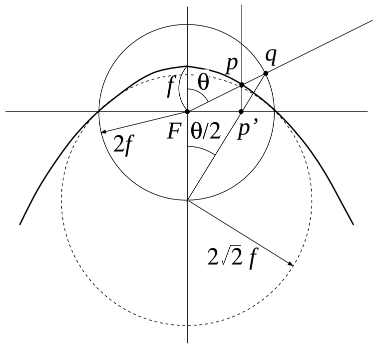
$$\tan\theta = \frac{2\tan\theta/2}{1 - \tan^2\theta/2}.$$

Hence,

$$d = 2f\tan\frac{\theta}{2}.$$

The imaging geometry is the same as the fisheye lens camera.

Stereoscopic projection an inversion

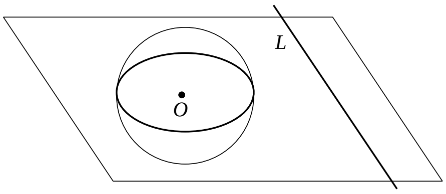


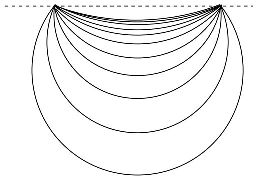
- Consider a hypothetical image sphere of radius $2f$ around the focus F of the parabolic mirror.
- Consider a hypothetical image plane passes through the focus F

Then,

- The image is a stereographic projection of the image sphere from its south pole.
- It is also an inversion of the image sphere with respect to an inversion sphere of radius $2\sqrt{2}f$ around the south pole of the image sphere.

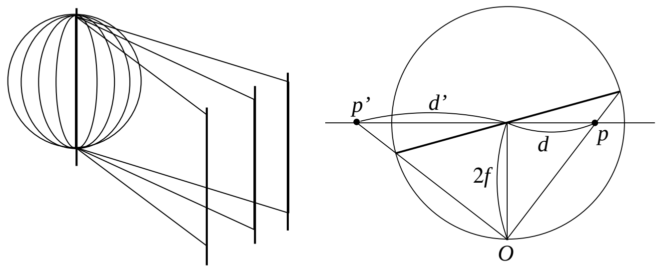
Projection of lines in the scene





- Lines in the scenes are mapped to great circles on the image sphere.
- Since the mapping from the image sphere to the image plane is conformal, great circles are mapped to *circles* in the image.
- Parallell lines in the scene are imaged as circles intersecting at common *vanishing points* corresponding to the common end points at infinity.

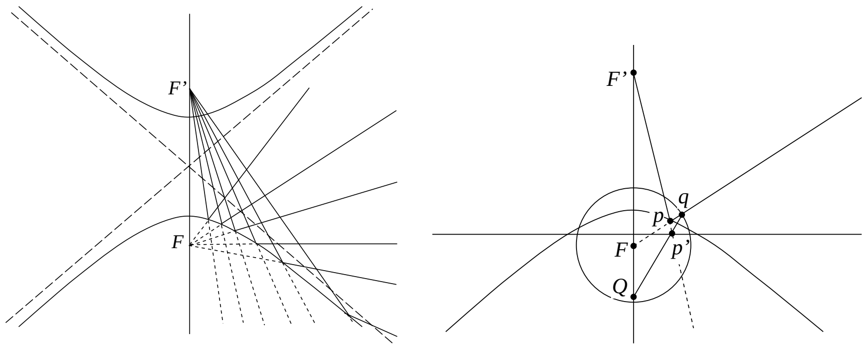
Vanishing points and the focal length



- The vanishing points indicate the 3-D direction of the parallel lines.
- The vanishing point pair is a stereoscopic projection of a diameter segment of the image sphere.
- The focal length is computed from the position of the vanishing points.
  - It is the *geometric mean* of the distances d and d' of the vanishing points from the principal point:

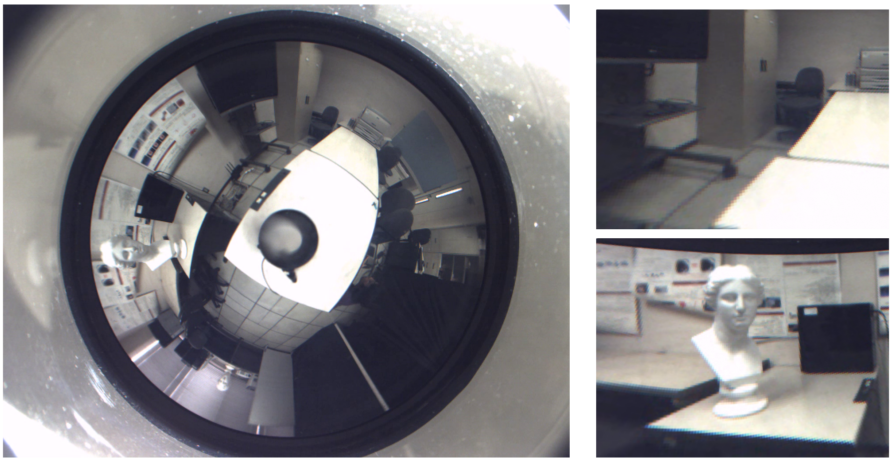
$$f = \frac{\sqrt{dd'}}{2}$$

Omnidirectional cameras using a hyperbolic mirror



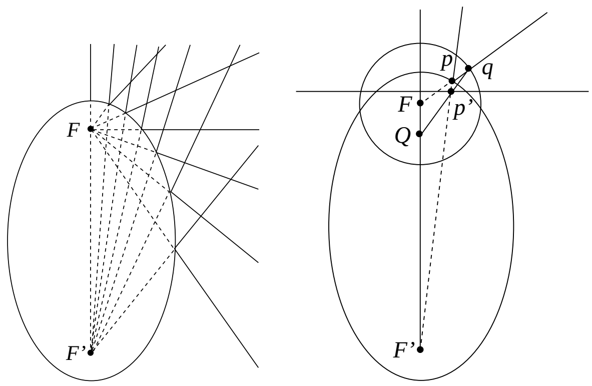
- Incoming rays toward one focus F are reflected so that they converge to the other focus F'.
- The observed image is a projection of the image sphere around F from a certain point onto the image plane placed in a certain position.
  - It is *not* a stereoscopic projection, so it is *not* a conformal mapping.
  - Hence, a line in the scene is imaged as an *ellipse* in the image.

Omnidirectional image example



An indoors scene image taken by an omnidirectional camera with a hyperbolic mirror, and perspectively transformed partial images.

Omnidirectional cameras using an elliptic mirror



- Incoming rays toward one focus F are reflected as if diverging from the other focus F'.
- The observed image is a projection of the image sphere around F from a certain point onto the image plane placed in a certain position.
  - It is *not* a stereoscopic projection, so it is *not* a conformal mapping.
  - Hence, a line in the scene is imaged as an *ellipse* in the image.
- For a given omnidirectional camera with an elliptic mirror, we can define an omnidirectional camera with a hyperbolic mirror such that the resulting images are the same.

Conclusion

Conclusions

- Is geometric algebra worth studying?
  - Yes, definitely. It is a well-defined and very inspiring mathematics.
- Does geometric algebra lead to new results of computer vision research?
  - We don't know. It depends on how it is used.
- Geometric algebra provides a very nice and very concise *description* of geometry.
  - However, it does not provide a means of *numerical computation*.
- For numerical computation, many researchers of geometric algebra offer software tools, inside of which high-dimensional matrix calculus is conducted.
  - Numerical computation by such software is not necessarily efficient. Research is going on to optimize the computation.
- Currently, three types of research papers are published in relation to geometric algebra:
  1. Propaganda papers, telling people how nice geometric algebra is.
  2. Geometric modeling demonstrations using software tools.
     - Scene modeling from video images taken by vehicle-mounted cameras.
     - Computer graphics applications.
  3. Techniques for improving the efficiency of geometric algebra software.