PAPER

Optimal Conic Fitting and Reliability Evaluation

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SUMMARY Introducing a mathematical model of image noise, we formalize the problem of fitting a conic to point data as statistical estimation. It is shown that the reliability of the fitted conic can be evaluated quantitatively in the form of the covariance tensor. We present a numerical scheme called *renormalization* for computing an optimal fit and at the same time evaluating its reliability. We also present a scheme for visualizing the reliability of the fit by means of the *primary deviation pair*. Our method is illustrated by showing simulations and real-image examples.

key words: conic fitting, reliability evaluation, statistical model of noise, image processing, statistical optimization

1. Introduction

Many industrial objects have circular and spherical shapes, which are projected as ellipses in their images. Therefore, ellipses, or *conics* in general, are very important image features in computer vision and robotics applications [6], [23], [24]. If a conic in an image is known to be a projection of a circle or an ellipse of known shape, its 3-D position can be computed by an analytical means (but not always uniquely) [12]. In order to do such an analysis, we must detect a conic by an edge operator as a sequence of pixels and fit a conic equation to the detected pixels.

Many conic fitting techniques have been proposed in the past: crude estimation based on the Hough transform [4], [7], [26], least-squares estimation with different parameterizations and criteria [2], [3], [16], [17], [21], [25], and various techniques based on edges, gray levels, heuristics, and other information [1], [5], [18], [20], [24] (see [22] for an overview). However, little attention has been paid to the statistical behavior of the image noise. An exception is Porrill [19], who incorporated a statistical consideration and applied an iterative filter, which he called the "extended Kalman filter". He pointed out the existence of statistical bias in the least-squares solution and proposed a correction scheme for removing it. Applying the traditional statistical approach to the curve fitting problem, Joseph [8] and Werman and Geyzel [27] analyzed the asymptotic behavior of the coefficients of the fitting equation for large noise in the limit of a large number of data.

In this paper, we formulate the conic fitting problem as *statistical estimation*, but our approach is opposite to that of [8] and [27]: we fix the number of data and do *perturbation analysis* for small noise. Introducing a mathematical model of image noise, we derive a *theoretically optimal* fitting scheme in the sense of *maximum likelihood estimation*. We also give an *explicit* expression that evaluates the reliability of the computed fit in quantitative terms and propose a scheme for *visualizing* the reliability of the fit by means of the *primary deviation pair*. Then, we present a simple computational scheme called *renormalization* for computing an optimal fit and at the same time evaluating its reliability. We illustrate our method by showing numerical simulations and real-image examples.

2. Conic Fitting

A *conic* is a curve on a two-dimensional plane whose equation has the form

$$Ax^2 + 2Bxy + Cy^2 + 2(Dx + Ey) + F = 0. \qquad (1)$$

If a point (x, y) is represented by a three-dimensional vector $\boldsymbol{x} = (x, y, 1)^\top$ (the superscript $\top$ denotes transpose), Eq. (1) can be written in the form

$$(\boldsymbol{x}, \boldsymbol{Q}\boldsymbol{x}) = 0, \qquad (2)$$

where $\boldsymbol{Q}$ is a three-dimensional symmetric matrix defined by

$$\boldsymbol{Q} = \begin{pmatrix} A & B & D \\ B & C & E \\ D & E & F \end{pmatrix}. \qquad (3)$$

In this paper, we denote the inner product of vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ by $(\boldsymbol{a}, \boldsymbol{b})$.

Let $\{(x_\alpha, y_\alpha)\}$, $\alpha = 1, ..., N$, be the points that are supposed to be on a conic. In real circumstances, digital images are not ideal, and image processing operations such as edge detection may not be accurate. We refer to such inaccuracy, irrespective of its sources, collectively as "image noise". Let $(\bar{x}_\alpha, \bar{y}_\alpha)$ be the true position of point (x_α, y_α), i.e., the position that would supposedly be observed if the image were ideal and the

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detection operation were accurate. We want to obtain a conic that passes through $(\bar{x}_\alpha, \bar{y}_\alpha)$.

Since the matrix $\boldsymbol{Q}$ in Eq. (2) is determined only up to scale, we adopt the normalization $\|\boldsymbol{Q}\| = 1$, where the matrix norm of $\boldsymbol{Q} = (Q_{ij})$ is defined by $\|\boldsymbol{Q}\| = \sqrt{\sum_{i,j=1}^{3} Q_{ij}^2}$. The conic fitting problem is formally stated as follows:

Problem: Estimate a symmetric matrix $\boldsymbol{Q}$ of unit norm such that

$$(\bar{\boldsymbol{x}}_\alpha, \boldsymbol{Q}\bar{\boldsymbol{x}}_\alpha) = 0, \quad \alpha = 1, ..., N, \tag{4}$$

from the data $\{\boldsymbol{x}_\alpha\}$, $\alpha = 1$, ..., N.

3. Optimal Estimation

We decompose $\boldsymbol{x}_\alpha$ into the form

$$\boldsymbol{x}_\alpha = \bar{\boldsymbol{x}}_\alpha + \Delta\boldsymbol{x}_\alpha, \tag{5}$$

and regard the noise term $\Delta\boldsymbol{x}_\alpha$ as an independent Gaussian random variable of mean $\boldsymbol{0}$ and covariance matrix

$$V[\boldsymbol{x}_\alpha] = E[\Delta\boldsymbol{x}_\alpha \Delta\boldsymbol{x}_\alpha^\top], \tag{6}$$

where $E[\,\cdot\,]$ denotes expectation. Since the third component of $\Delta\boldsymbol{x}_\alpha$ is always zero, $V[\boldsymbol{x}_\alpha]$ is a singular matrix of rank 2.

It can be shown that if the image noise is small and the product of two Gaussian random variables are approximated to be Gaussian, the optimal estimate of $\boldsymbol{Q}$ can be obtained as the solution of the minimization

$$J[\boldsymbol{Q}] = \sum_{\alpha=1}^{N} \frac{(\boldsymbol{x}_\alpha \otimes \boldsymbol{x}_\alpha - V[\boldsymbol{x}_\alpha]; \boldsymbol{Q})^2}{4(\boldsymbol{x}_\alpha, \boldsymbol{Q}V[\boldsymbol{x}_\alpha]\boldsymbol{Q}\boldsymbol{x}_\alpha) + 2(V[\boldsymbol{x}_\alpha]\boldsymbol{Q}; \boldsymbol{Q}V[\boldsymbol{x}_\alpha])} \to \min, \tag{7}$$

under the constraint $\|\boldsymbol{Q}\| = 1$ [11] (see Appendix). In this paper, we define the inner product of matrices $\boldsymbol{A} = (A_{ij})$ and $\boldsymbol{B} = (B_{ij})$ by $(\boldsymbol{A}; \boldsymbol{B}) = \sum_{i,j=1}^{3} A_{ij}B_{ij}$. The symbol $\otimes$ denotes the tensor product.

Let $\hat{\boldsymbol{Q}}$ be the solution of the minimization (7), and write

$$\hat{\boldsymbol{Q}} = \bar{\boldsymbol{Q}} + \Delta\boldsymbol{Q}, \tag{8}$$

where $\bar{\boldsymbol{Q}}$ is the true value of $\boldsymbol{Q}$. The reliability of $\hat{\boldsymbol{Q}}$ is measured by its *covariance tensor*

$$\mathcal{V}[\hat{\boldsymbol{Q}}] = E[\Delta\boldsymbol{Q} \otimes \Delta\boldsymbol{Q}]. \tag{9}$$

It can be shown that this covariance tensor has the following form [11]:

$$\mathcal{V}[\hat{\boldsymbol{Q}}] = \left(\sum_{\alpha=1}^{N} \frac{\mathcal{P}(\bar{\boldsymbol{x}}_\alpha \otimes \bar{\boldsymbol{x}}_\alpha) \otimes \mathcal{P}(\bar{\boldsymbol{x}}_\alpha \otimes \bar{\boldsymbol{x}}_\alpha)}{4(\bar{\boldsymbol{x}}_\alpha, \bar{\boldsymbol{Q}}V[\boldsymbol{x}_\alpha]\bar{\boldsymbol{Q}}\bar{\boldsymbol{x}}_\alpha) + 2(V[\boldsymbol{x}_\alpha]\bar{\boldsymbol{Q}}; \bar{\boldsymbol{Q}}V[\boldsymbol{x}_\alpha])} \right)^{-}. \tag{10}$$

Here, the superscript "−" denotes the (*Moore-Penrose*) *generalized inverse*, and $\mathcal{P} = (P_{ijkl})$ is the *projection tensor* defined by

$$P_{ijkl} = \delta_{ik}\delta_{jl} - \bar{Q}_{ij}\bar{Q}_{kl}, \tag{11}$$

where δ_{ij} is the *Kronecker delta*, taking value 1 for $i = j$ and value 0 otherwise. It can be shown that Eq. (10) gives a theoretical bound on attainable accuracy called the *Cramer-Rao lower bound* [11].

4. Least-Squares Approximation

We decompose the covariance matrix $V[\boldsymbol{x}_\alpha]$ into the form

$$V[\boldsymbol{x}_\alpha] = \epsilon^2 V_0[\boldsymbol{x}_\alpha]. \tag{12}$$

The constant ϵ indicates the average magnitude of the image noise; we call it the *noise level*. The matrix $V_0[\boldsymbol{x}_\alpha]$ indicates in which orientation the deviation is likely to occur; we call it the *normalized covariance matrix*. In many circumstances, the qualitative characteristics of image noise, such as homogeneity/inhomogeneity, isotropy/anisotropy, and their relative degrees, can be discerned relatively easily from the characteristics of the imaging device and the image processing algorithm, whereas its absolute magnitude is very difficult to predict a priori. Here, we assume that the covariance matrix $V[\boldsymbol{x}_\alpha]$ is known only *up to scale*: $V_0[\boldsymbol{x}_\alpha]$ is known, but ϵ is unknown.

If the denominator in Eq. (7) is replaced by a constant, we obtain the *least-squares approximation*

$$J[\boldsymbol{Q}] = \sum_{\alpha=1}^{N} (\boldsymbol{Q}; \mathcal{M}^*\boldsymbol{Q}) \to \min. \tag{13}$$

Here, $\mathcal{M}^*$ is the *effective moment tensor* defined by

$$\mathcal{M}^* = \frac{1}{N}\sum_{\alpha=1}^{N} W_\alpha(\boldsymbol{x}_\alpha \otimes \boldsymbol{x}_\alpha - \epsilon^2 V_0[\boldsymbol{x}_\alpha]) \otimes (\boldsymbol{x}_\alpha \otimes \boldsymbol{x}_\alpha - \epsilon^2 V_0[\boldsymbol{x}_\alpha]), \tag{14}$$

$$W_\alpha = \frac{1}{4(\boldsymbol{x}_\alpha, \boldsymbol{Q}^*V_0[\boldsymbol{x}_\alpha]\boldsymbol{Q}^*\boldsymbol{x}_\alpha) + 2\epsilon^2(V_0[\boldsymbol{x}_\alpha]\boldsymbol{Q}^*; \boldsymbol{Q}^*V_0[\boldsymbol{x}_\alpha])}, \tag{15}$$

where $\boldsymbol{Q}^*$ is an appropriate estimate of $\boldsymbol{Q}$. If we let $\epsilon^2 = 0$ in Eqs. (14) and (15), Eq. (13) reduces to the widely used *least-squares method* [9].

The solution of the minimization (13) is obtained as the *eigenmatrix* of norm 1 of tensor $\mathcal{M}^*$ for the smallest eigenvalue [9], where we say that matrix $\boldsymbol{A}$ is an eigenmatrix of tensor $\mathcal{T}$ for eigenvalue λ if $\mathcal{T}\boldsymbol{A} = \lambda\boldsymbol{A}$ (the product $\mathcal{T}\boldsymbol{A}$ of a tensor $\mathcal{T} = (T_{ijkl})$ and a matrix $\boldsymbol{A} = (A_{ij})$ is a matrix whose (ij) element is

$\sum_{k,l=1}^{3} T_{ijkl}A_{kl}$). In order to compute the eigenmatrix $\boldsymbol{Q}$ of tensor $\mathcal{M}^*$, we identify the matrix $\boldsymbol{Q} = (Q_{ij})$ and the tensor $\mathcal{M}^* = (M^*_{ijkl})$ with the following six-dimensional vector $\boldsymbol{q}$ and six-dimensional matrix $\boldsymbol{M}^*$, respectively:

$$\boldsymbol{q} = (Q_{11}, Q_{22}, Q_{33}, \sqrt{2}Q_{23}, \sqrt{2}Q_{31}, \sqrt{2}Q_{12})^\top, \quad (16)$$

$$\boldsymbol{M}^* = \begin{pmatrix} M^*_{1111} & M^*_{1122} & M^*_{1133} & \sqrt{2}M^*_{1123} & \sqrt{2}M^*_{1131} & \sqrt{2}M^*_{1112} \\ M^*_{2211} & M^*_{2222} & M^*_{2233} & \sqrt{2}M^*_{2223} & \sqrt{2}M^*_{2231} & \sqrt{2}M^*_{2212} \\ M^*_{3311} & M^*_{3322} & M^*_{3333} & \sqrt{2}M^*_{3323} & \sqrt{2}M^*_{3331} & \sqrt{2}M^*_{3312} \\ \sqrt{2}M^*_{2311} & \sqrt{2}M^*_{2322} & \sqrt{2}M^*_{2333} & 2M^*_{2323} & 2M^*_{2331} & 2M^*_{2312} \\ \sqrt{2}M^*_{3111} & \sqrt{2}M^*_{3122} & \sqrt{2}M^*_{3133} & 2M^*_{3123} & 2M^*_{3131} & 2M^*_{3112} \\ \sqrt{2}M^*_{1211} & \sqrt{2}M^*_{1222} & \sqrt{2}M^*_{1233} & 2M^*_{1223} & 2M^*_{1231} & 2M^*_{1212} \end{pmatrix}. \quad (17)$$

Then, $\boldsymbol{Q}$ is an eigenmatrix of tensor $\mathcal{M}^*$ for eigenvalue λ if and only if $\boldsymbol{q}$ is an eigenvector of matrix $\boldsymbol{M}^*$ for eigenvalue λ.

5. Unbiased Estimation

It can be shown that the solution of the least-squares approximation is statistically biased whatever weights W_α are used [9], [10]. In fact, define the *moment tensor*

$$\mathcal{M} = \frac{1}{N}\sum_{\alpha=1}^{N} W_\alpha \boldsymbol{x}_\alpha \otimes \boldsymbol{x}_\alpha \otimes \boldsymbol{x}_\alpha \otimes \boldsymbol{x}_\alpha, \quad (18)$$

and let $\bar{\mathcal{M}}$ be the unperturbed moment tensor obtained by replacing $\boldsymbol{x}_\alpha$ by $\bar{\boldsymbol{x}}_\alpha$. Equation (4) implies that $\boldsymbol{Q}$ is the eigenmatrix of $\bar{\mathcal{M}}$ for eigenvalue 0. However, the tensor $\mathcal{M}^*$ defined by Eq. (14) has expectation $E[\mathcal{M}^*] = \bar{\mathcal{M}} + O(\epsilon^2)$. Hence, the expectation of its eigenmatrix is biased from its true value by $O(\epsilon^2)$ according to the *perturbation theorem* [9]. This bias can be removed by the following procedure.

Define tensors $\mathcal{N}^{(1)} = (N^{(1)}_{ijkl})$ and $\mathcal{N}^{(2)} = (N^{(2)}_{ijkl})$ by

$$\begin{aligned} N^{(1)}_{ijkl} = \frac{1}{N}\sum_{\alpha=1}^{N} W_\alpha \big(& V_0[\boldsymbol{x}_\alpha]_{ij} x_{\alpha(k)} x_{\alpha(l)} \\ & + V_0[\boldsymbol{x}_\alpha]_{ik} x_{\alpha(j)} x_{\alpha(l)} + V_0[\boldsymbol{x}_\alpha]_{il} x_{\alpha(j)} x_{\alpha(k)} \\ & + V_0[\boldsymbol{x}_\alpha]_{jk} x_{\alpha(i)} x_{\alpha(l)} + V_0[\boldsymbol{x}_\alpha]_{jl} x_{\alpha(i)} x_{\alpha(k)} \\ & + V_0[\boldsymbol{x}_\alpha]_{kl} x_{\alpha(i)} x_{\alpha(j)} \big), \end{aligned} \quad (19)$$

$$\begin{aligned} N^{(2)}_{ijkl} = \frac{1}{N}\sum_{\alpha=1}^{N} W_\alpha \big(& V_0[\boldsymbol{x}_\alpha]_{ij} V_0[\boldsymbol{x}_\alpha]_{kl} \\ & + V_0[\boldsymbol{x}_\alpha]_{ik} V_0[\boldsymbol{x}_\alpha]_{jl} + V_0[\boldsymbol{x}_\alpha]_{il} V_0[\boldsymbol{x}_\alpha]_{jk} \big). \end{aligned} \quad (20)$$

Let $\bar{\mathcal{N}}^{(1)}$ be the unperturbed value of $\mathcal{N}^{(1)}$ obtained by replacing $\boldsymbol{x}_\alpha$ by $\bar{\boldsymbol{x}}_\alpha$ in Eq. (19). Then, we obtain the following relation:

$$E[\mathcal{M}] = \bar{\mathcal{M}} + \epsilon^2 \bar{\mathcal{N}}^{(1)} + \epsilon^4 \mathcal{N}^{(2)}. \quad (21)$$

It follows that if we define the *unbiased moment tensor*

$$\hat{\mathcal{M}} = \mathcal{M} - \epsilon^2 \mathcal{N}^{(1)} + \epsilon^4 \mathcal{N}^{(2)}, \quad (22)$$

we have $E[\hat{\mathcal{M}}] = \bar{\mathcal{M}}$. Hence, an unbiased estimator of $\boldsymbol{Q}$ is obtained by the optimization

$$\hat{J}[\boldsymbol{Q}] = (\boldsymbol{Q}; \hat{\mathcal{M}}\boldsymbol{Q}) \to \min. \quad (23)$$

The solution is obtained as the eigenmatrix of $\hat{\mathcal{M}}$ of norm 1 for the smallest eigenvalue [9].

6. Renormalization

In order to compute the unbiased moment tensor $\hat{\mathcal{M}}$, we need to estimate the noise level ϵ precisely, which is very difficult as we mentioned earlier. If ϵ is underestimated, the bias still remains, while if it is overestimated, the bias with opposite sign occurs. This difficulty can be avoided by iteratively estimating ϵ so that the smallest eigenvalue of $\hat{\mathcal{M}}$ becomes 0. This procedure, which we call the (second order) *renormalization*, is given as follows [10], [13], [15]:

1. Let $c = 0$ and $W_\alpha = 1$, $\alpha = 1, \ldots, N$.
2. Compute the tensors $\mathcal{M}$, $\mathcal{N}^{(1)}$, and $\mathcal{N}^{(2)}$ defined by Eqs. (18), (19), and (20), respectively.
3. Compute the smallest eigenvalue λ of the tensor
$$\hat{\mathcal{M}} = \mathcal{M} - c\mathcal{N}^{(1)} + c^2 \mathcal{N}^{(2)}, \quad (24)$$
and the corresponding eigenmatrix $\boldsymbol{Q}$ of norm 1.
4. If $\lambda \approx 0$, return $\boldsymbol{Q}$, c, and $\hat{\mathcal{M}}$. Else, update c and W_α as follows:
$$\begin{aligned} D = & \Big((\boldsymbol{Q}; \mathcal{N}^{(1)}\boldsymbol{Q}) - 2c(\boldsymbol{Q}; \mathcal{N}^{(2)}\boldsymbol{Q}) \Big)^2 \\ & - 4\lambda(\boldsymbol{Q}; \mathcal{N}^{(2)}\boldsymbol{Q}), \end{aligned} \quad (25)$$
$$\begin{aligned} & \text{If} \quad D \geqq 0, \\ & \quad c \leftarrow c + \frac{(\boldsymbol{Q}; \mathcal{N}^{(1)}\boldsymbol{Q}) - 2c(\boldsymbol{Q}; \mathcal{N}^{(2)}\boldsymbol{Q}) - \sqrt{D}}{2(\boldsymbol{Q}; \mathcal{N}^{(2)}\boldsymbol{Q})}, \\ & \text{If} \quad D < 0, \quad c \leftarrow c + \frac{\lambda}{(\boldsymbol{Q}; \mathcal{N}^{(1)}\boldsymbol{Q})}, \end{aligned} \quad (26)$$
$$W_\alpha \leftarrow \frac{1}{4(\boldsymbol{x}_\alpha, \boldsymbol{Q}V_0[\boldsymbol{x}_\alpha]\boldsymbol{Q}\boldsymbol{x}_\alpha) + 2c(V_0[\boldsymbol{x}_\alpha]\boldsymbol{Q}; \boldsymbol{Q}V_0[\boldsymbol{x}_\alpha])}. \quad (27)$$

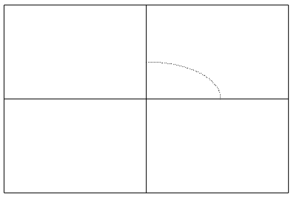
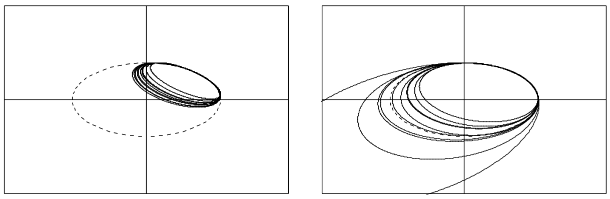


Fig. 1 Equidistant points on an ellipse in the first quadrant.



(a) (b)

Fig. 2 (a) Ten fits obtained by the least-squares method. (b) Corresponding fits obtained by renormalization.

5. Go back to Step 2.

Let $\hat{\boldsymbol{Q}}$ be the returned value of $\boldsymbol{Q}$. An unbiased estimator of the squared noise level ϵ^2 is obtained in the form

$$\hat{\epsilon}^2 = \frac{c}{1 - 5/N}. \tag{28}$$

This is a consequence of the fact that $N\hat{J}[\hat{\boldsymbol{Q}}]/\epsilon^2$ is subject to a χ^2 distribution with $N-5$ degrees of freedom in the first order [11], where $\hat{J}[\hat{\boldsymbol{Q}}]$ is the residual of the optimization (23). The covariance tensor of the estimator $\hat{\boldsymbol{Q}}$ is estimated by

$$\mathcal{V}[\hat{\boldsymbol{Q}}] = \frac{\hat{\epsilon}^2}{N}\left(\hat{\mathcal{M}}\right)_5^-, \tag{29}$$

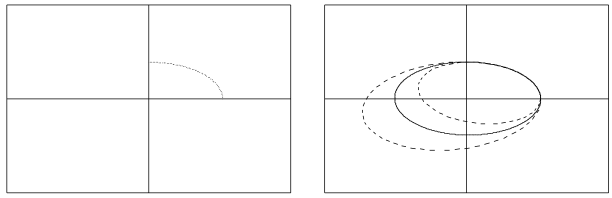
where $(\,\cdot\,)_5^-$ denotes the generalized inverse computed after projecting the tensor onto a tensor of rank 5 by ignoring the smallest eigenvalue [11], [13]. This operation is necessary because the smallest eigenvalue of $\hat{\mathcal{M}}$ may not be strictly 0 if the renormalization iterations are prematurely terminated.

The renormalization procedure described above is obtained by extending the prototype proposed in [10] in such a way that error terms of up to the second order are compensated for. However, the difference due to this extension is very small, so in actual computation it suffices to use Eq. (26), ignoring the term $c^2\mathcal{N}^{(2)}$ in Eq. (24).

7. Primary Deviation Pair

We can decompose the covariance tensor $\mathcal{V}[\hat{\boldsymbol{Q}}]$ into the following form by applying the *spectral decomposition* [9]:

$$\mathcal{V}[\hat{\boldsymbol{Q}}] = \sum_{i=1}^{5} \lambda_i \boldsymbol{U}_i \otimes \boldsymbol{U}_i^\top, \tag{30}$$



(a) (b)

Fig. 3 (a) Points perturbed by noise. (b) An optimal fit (solid line) and its primary deviation pair (dashed lines).

Here, λ_i is the ith largest eigenvalue of the tensor $\mathcal{V}[\hat{\boldsymbol{Q}}]$; $\boldsymbol{U}_i$ is the corresponding eigenmatrix of unit norm. The matrix $\boldsymbol{U}_1$ indicates the most likely mode of deviation; λ_1 is the variance in that mode. It follows that the reliability of the estimator $\hat{\boldsymbol{Q}}$ can be visualized by displaying the two conics represented by

$$\begin{aligned} \boldsymbol{Q}^+ &= N[\hat{\boldsymbol{Q}} + \sqrt{\lambda_1}\boldsymbol{U}_1], \\ \boldsymbol{Q}^- &= N[\hat{\boldsymbol{Q}} - \sqrt{\lambda_1}\boldsymbol{U}_1], \end{aligned} \tag{31}$$

where $N[\,\cdot\,]$ denotes normalization into a matrix of unit norm. We call these two conics the *primary deviation pair* [13]–[15].

8. Examples

Figure 1 shows sixty equidistant points on an ellipse in the first quadrant. The major and minor radii are assumed to be 100 pixels and 50 pixels, respectively. We added Gaussian random noise of mean 0 and standard deviation 0.5 pixels to the x and y coordinates of each point independently. Figure 2(a) shows ten fits computed by the least-squares method (by using the weight W_α given by Eq. (15) for the true conic), each time using different noise; the true conic is drawn in a dashed line. The existence of statistical bias is evident. Figure 2(b) shows corresponding optimal fits computed by renormalization. We can see that the statistical bias has been removed. Figure 3(a) shows one instance of perturbed points; Figure 3(b) shows a fitted conic and its primary deviation pair. We can see that the primary deviation pair well characterizes the random deviations shown in Fig. 2(b).

Figure 4(a) is part of an edge image obtained by applying an edge operator to a real image. A conic is optimally fitted to one edge segment that constitutes a conic and superimposed on the original real image in a solid line (Fig. 4(b)); its primary deviation pair is drawn in dashed lines. Figure 5 shows another example. As we can see, not only an optimal fit can be obtained but its reliability can also be visualized without any knowledge of the image noise. We can also see that the reliability rapidly decreases as the length of the conic edge segment decreases.

9. Concluding Remarks

Introducing a mathematical model of image noise, we have formalized the problem of fitting a conic to point data as statistical estimation. We have shown that the reliability of the fitted conic can be evaluated quantitatively in the form of the covariance tensor and presented a numerical scheme called *renormalization* for computing an optimal fit and at the same time evaluating its reliability. We have also presented a scheme for visualizing the reliability of the fit by means of the *primary deviation pair* and shown simulations and real-image examples to illustrate our method.

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Appendix: Optimal Estimation

Define a matrix

$$\boldsymbol{X}_\alpha = \boldsymbol{x}_\alpha \otimes \boldsymbol{x}_\alpha, \tag{A·1}$$

and let $\bar{\boldsymbol{X}}_\alpha$ be the unperturbed value of $\boldsymbol{X}_\alpha$ obtained by replacing $\boldsymbol{x}_\alpha$ by $\bar{\boldsymbol{x}}_\alpha$. Then, Eq. (4) can be written in the form

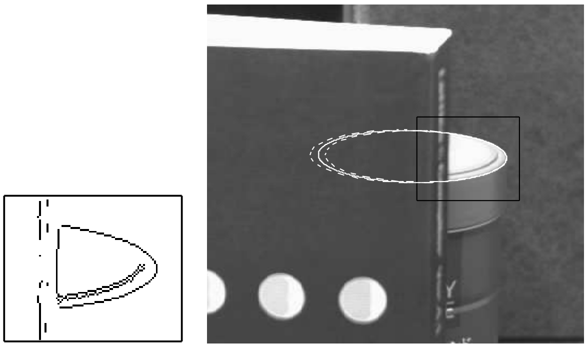
$$(\bar{\boldsymbol{X}}_\alpha; \boldsymbol{Q}) = 0. \tag{A·2}$$

Substituting Eq. (5) into Eq. (A·1), we can decompose Eq. (A·1) into the form

$$\begin{aligned}\boldsymbol{X}_\alpha &= (\bar{\boldsymbol{x}}_\alpha + \Delta\boldsymbol{x}_\alpha) \otimes (\bar{\boldsymbol{x}}_\alpha + \Delta\boldsymbol{x}_\alpha) \\ &= \bar{\boldsymbol{X}}_\alpha + \Delta\boldsymbol{x}_\alpha \otimes \bar{\boldsymbol{x}}_\alpha + \bar{\boldsymbol{x}}_\alpha \otimes \Delta\boldsymbol{x}_\alpha \\ &\quad + \Delta\boldsymbol{x}_\alpha \otimes \Delta\boldsymbol{x}_\alpha.\end{aligned} \tag{A·3}$$

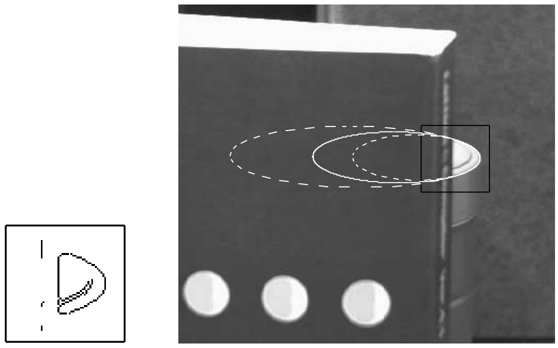
Since $E[\Delta\boldsymbol{x}_\alpha]$=0, we see that

$$E[\boldsymbol{X}_\alpha] = \bar{\boldsymbol{X}}_\alpha + V[\boldsymbol{x}_\alpha]. \tag{A·4}$$



(a) (b)

Fig. 4 (a) An edge image. (b) An optimally fitted conic and its primary deviation pair.



(a) (b)

Fig. 5 (a) An edge image. (b) An optimally fitted conic and its primary deviation pair.

Define the *effective value* $\boldsymbol{X}^*_\alpha$ of $\boldsymbol{X}_\alpha$ by

$$\boldsymbol{X}^*_\alpha = \boldsymbol{X}_\alpha - V[\boldsymbol{x}_\alpha]. \tag{A·5}$$

If we write $\boldsymbol{X}^*_\alpha = \bar{\boldsymbol{X}}_\alpha + \Delta\boldsymbol{X}_\alpha$, we can regard $\Delta\boldsymbol{X}_\alpha$ as the following matrix random variable of mean $\boldsymbol{O}$:

$$\begin{aligned}\Delta\boldsymbol{X}_\alpha &= \boldsymbol{X}^*_\alpha - \bar{\boldsymbol{X}}_\alpha \\ &= \Delta\boldsymbol{x}_\alpha \otimes \bar{\boldsymbol{x}}_\alpha + \bar{\boldsymbol{x}}_\alpha \otimes \Delta\boldsymbol{x}_\alpha + \Delta\boldsymbol{x}_\alpha \otimes \Delta\boldsymbol{x}_\alpha \\ &\quad - V[\boldsymbol{x}_\alpha]. \end{aligned}\tag{A·6}$$

If the ith components of $\bar{\boldsymbol{x}}_\alpha$ and $\Delta\boldsymbol{x}_\alpha$ are written as $\bar{\boldsymbol{x}}_{\alpha(i)}$ and $\Delta\boldsymbol{x}_{\alpha(i)}$, respectively, the covariance tensor $\mathcal{V}[\boldsymbol{X}_\alpha]$ has the following $(ijkl)$ element:

$$\begin{aligned} E[\Delta\boldsymbol{X}_\alpha \otimes \Delta\boldsymbol{X}_\alpha]_{ijkl} &= E[\Delta x_{\alpha(j)}\Delta x_{\alpha(k)}]\bar{x}_{\alpha(i)}\bar{x}_{\alpha(l)} \\ &\quad + E[\Delta x_{\alpha(j)}\Delta x_{\alpha(l)}]\bar{x}_{\alpha(i)}\bar{x}_{\alpha(k)} \\ &\quad + E[\Delta x_{\alpha(i)}\Delta x_{\alpha(k)}]\bar{x}_{\alpha(j)}\bar{x}_{\alpha(l)} \\ &\quad + E[\Delta x_{\alpha(i)}\Delta x_{\alpha(l)}]\bar{x}_{\alpha(j)}\bar{x}_{\alpha(k)} \\ &\quad - E[\Delta x_{\alpha(i)}\Delta x_{\alpha(j)}]V[\boldsymbol{x}_\alpha]_{kl} \\ &\quad - E[\Delta x_{\alpha(k)}\Delta x_{\alpha(l)}]V[\boldsymbol{x}_\alpha]_{ij} \\ &\quad + E[\Delta x_{\alpha(i)}\Delta x_{\alpha(j)}\Delta x_{\alpha(k)}\Delta x_{\alpha(l)}] \\ &\quad + V[\boldsymbol{x}_\alpha]_{ij}V[\boldsymbol{x}_\alpha]_{kl} \\ &= V[\boldsymbol{x}_\alpha]_{jl}\bar{x}_{\alpha(i)}\bar{x}_{\alpha(k)} + V[\boldsymbol{x}_\alpha]_{jk}\bar{x}_{\alpha(i)}\bar{x}_{\alpha(l)} \\ &\quad + V[\boldsymbol{x}_\alpha]_{il}\bar{x}_{\alpha(j)}\bar{x}_{\alpha(k)} + V[\boldsymbol{x}_\alpha]_{ik}\bar{x}_{\alpha(j)}\bar{x}_{\alpha(l)} \\ &\quad + V[\boldsymbol{x}_\alpha]_{ik}V[\boldsymbol{x}_\alpha]_{jl} + V[\boldsymbol{x}_\alpha]_{il}V[\boldsymbol{x}_\alpha]_{jk}. \end{aligned}\tag{A·7}$$

Here, we have used the following identity for a Gaussian random variable $\boldsymbol{v} = (v_i)$ of mean $\boldsymbol{0}$ and covariance matrix $\boldsymbol{\Sigma} = (\Sigma_{ij})$ [11]:

$$E[v_iv_jv_kv_l] = \Sigma_{ij}\Sigma_{kl} + \Sigma_{ik}\Sigma_{jl} + \Sigma_{il}\Sigma_{jk}. \tag{A·8}$$

It can be shown that the optimal estimator of $\boldsymbol{Q}$ is obtained by the minimization [11]

$$J[\boldsymbol{Q}] = \sum_{\alpha=1}^{N} \frac{(\boldsymbol{X}^*_\alpha; \boldsymbol{Q})^2}{(\boldsymbol{Q}; \mathcal{V}[\boldsymbol{X}_\alpha]\boldsymbol{Q})} \to \min. \tag{A·9}$$

From Eq. (A·7), we obtain

$$\begin{aligned}(\boldsymbol{Q}; \mathcal{V}[\boldsymbol{X}_\alpha]\boldsymbol{Q}) &= 4(\bar{\boldsymbol{x}}_\alpha, \boldsymbol{Q}V[\boldsymbol{x}_\alpha]\bar{\boldsymbol{x}}_\alpha) \\ &\quad +2(V[\boldsymbol{x}_\alpha]\boldsymbol{Q}; \boldsymbol{Q}V[\boldsymbol{x}_\alpha]). \end{aligned}\tag{A·10}$$

Approximating $\bar{\boldsymbol{x}}_\alpha$ by data $\boldsymbol{x}_\alpha$, we obtain Eq. (7).

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