LETTER

Automatic Recognition of Regular Figures by Geometric AIC *

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SUMMARY We implement a graphical interface that automatically transforms a figure input by a mouse into a regular figure which the system infers is the closest to the input. The difficulty lies in the fact that the classes into which the input is to be classified have inclusion relations, which prohibit us from using a simple distance criterion. In this letter, we show that this problem can be resolved by introducing the *geometric AIC*.
key words: geometric AIC, model selection, CAD, graphics interface, pattern recognition

1. Introduction

A mouse is one of the most fundamental interfaces for generating figures interactively. One problem with it is generation of *regular* figures. Design of industrial objects requires many kinds or regularity such as orthogonality and parallelism. However, the input figure need not satisfy the required regularity if the mouse is manipulated by a human. In many drawing tools, users are required to choose a specific mode of regularity (e.g., mode for rectangles) from a menu beforehand or enforce a specific regularity by inputting a command afterward. Is it not possible to automate this process? For example, if a user inputs an approximate rectangle, is it not possible for the computer to automatically infers the intended shape and correct the input figure into the inferred shape?

This appears simple at first sight. For example, we introduce some distance measure that describes dissimilarity between two figures. We prepare candidate classes of regular figures such as the class of rectangles and the class of squares. Given an input figure, we choose from each class the closest figure to the input in the distance measure we defined. Finally, we choose the one that has the smallest distance among them.

This paradigm has a fatal flaw. This is because classes of regular figures have *inclusion relations*. For example, the class of squares is a subset of the class of rectangles. It follows that the distance from any figure to the closest square is always no more than the distance to the closest rectangle. This means that squares are never chosen. In general, classes that are included in other classes are never chosen whatever distance measure is used (Fig. 1).

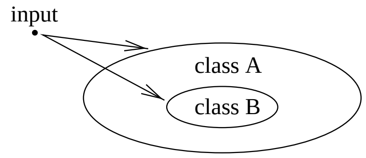

Fig. 1 A class included in another is not chosen.

In *pattern recognition*, it is tacitly assumed that the classes into which an input is to be classified are *disjoint*. One solution to the problem of classes with inclusion is artificial separation of the classes. For example, we may introduce an empirical threshold ϵ and decide that a rectangle is a square if the ratio of the lengths of adjacent sides are between $1-\epsilon$ and $1+\epsilon$, thereby separating the class of squares from the class of rectangles. However, the inclusion relation is one of the most important bases of geometric reasoning; its artificial disruption might cause difficulties in automated reasoning. Moreover, how can we determine the threshold ϵ? There exists no guiding principle for its determination.

In statistics, a well known criterion for selecting a reasonable model is the *AIC* [1]. However, inferences in statistics are usually formulated as *estimating the parameters of the distribution from which the data are drawn*. It follows that it is difficult to apply the AIC to the problem we are now concerned. However, we can generalize the principle that underlies the AIC to geometric inference. The resulting criterion is called the *geometric AIC* [2], [3]. In this letter, we present a scheme for classifying regular figures without introducing any empirical thresholds.

2. Classification of Quadrilaterals

Consider trapezoids, parallelograms, diamonds (rhombuses), rectangles, and squares as typical examples of regular figures (Fig. 2). We represent a point (x, y) in two dimensions by a three-dimensional vector $\boldsymbol{x} = (x, y, 1)^\top$ ($\top$ denotes transpose). Consider a quadrilateral defined by connecting four points $\boldsymbol{x}_1$, $\boldsymbol{x}_2$, $\boldsymbol{x}_3$, and $\boldsymbol{x}_4$ in that order.

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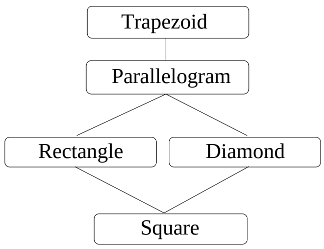

Fig. 2 Class inclusion relations for quadrilaterals.

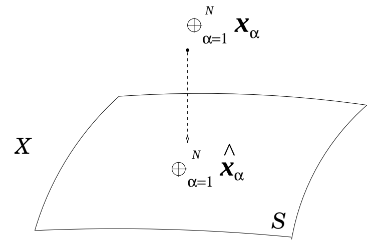

Fig. 3 Orthogonal projection onto the model.

Let us call the necessary and sufficient condition for a figure to be regular the *constraint equation(s)* for the regularity. For the regular figures listed in Fig. 2, the constraint equations are as follows. Here, $|\cdot,\cdot,|$ denotes scalar triple product, and $(\cdot,\cdot)$ denotes inner product; we define $\boldsymbol{k}=(0,0,1)^\top$.

Trapezoids. At least two sides are parallel:

$$|\boldsymbol{x}_2-\boldsymbol{x}_1,\boldsymbol{x}_4-\boldsymbol{x}_3,\boldsymbol{k}|\cdot|\boldsymbol{x}_3-\boldsymbol{x}_2,\boldsymbol{x}_1-\boldsymbol{x}_4,\boldsymbol{k}|=0.$$

Parallelograms. The two pairs of sides are parallel:

$$|\boldsymbol{x}_2-\boldsymbol{x}_1,\boldsymbol{x}_4-\boldsymbol{x}_3,\boldsymbol{k}|=0,$$
$$|\boldsymbol{x}_3-\boldsymbol{x}_2,\boldsymbol{x}_1-\boldsymbol{x}_4,\boldsymbol{k}|=0.$$

Rectangles. The two pairs of sides are parallel, and adjacent sides are orthogonal:

$$|\boldsymbol{x}_2-\boldsymbol{x}_1,\boldsymbol{x}_4-\boldsymbol{x}_3,\boldsymbol{k}|=0,$$
$$|\boldsymbol{x}_3-\boldsymbol{x}_2,\boldsymbol{x}_1-\boldsymbol{x}_4,\boldsymbol{k}|=0,$$
$$(\boldsymbol{x}_3-\boldsymbol{x}_2,\boldsymbol{x}_2-\boldsymbol{x}_1)=0.$$

Diamonds (rhombuses). The two pairs of sides are parallel, and the two diagonals are orthogonal:

$$|\boldsymbol{x}_2-\boldsymbol{x}_1,\boldsymbol{x}_4-\boldsymbol{x}_3,\boldsymbol{k}|=0,$$
$$|\boldsymbol{x}_3-\boldsymbol{x}_2,\boldsymbol{x}_1-\boldsymbol{x}_4,\boldsymbol{k}|=0,$$
$$(\boldsymbol{x}_3-\boldsymbol{x}_1,\boldsymbol{x}_4-\boldsymbol{x}_2)=0.$$

Squares. The two pairs of sides are parallel, adjacent sides are orthogonal, and the two diagonals are orthogonal:

$$|\boldsymbol{x}_2-\boldsymbol{x}_1,\boldsymbol{x}_4-\boldsymbol{x}_3,\boldsymbol{k}|=0,$$
$$|\boldsymbol{x}_3-\boldsymbol{x}_2,\boldsymbol{x}_1-\boldsymbol{x}_4,\boldsymbol{k}|=0,$$
$$(\boldsymbol{x}_3-\boldsymbol{x}_2,\boldsymbol{x}_2-\boldsymbol{x}_1)=0,$$
$$(\boldsymbol{x}_3-\boldsymbol{x}_1,\boldsymbol{x}_4-\boldsymbol{x}_2)=0.$$

3. Geometric AIC

Consider a figure with N vertices $\boldsymbol{x}_1$, ..., $\boldsymbol{x}_N$ with L constraint equations. By definition, the third components of vectors $\boldsymbol{x}_1$, ..., $\boldsymbol{x}_N$ are all 1. Hence, they have $2N$ degrees of freedom in all, so the direct sum $\bigoplus_{\alpha=1}^N \boldsymbol{x}_\alpha$ can be identified with a point in a $2N$-dimensional space $\mathcal{X}$, in which L constraint equations define a $(2N-L)$-dimensional manifold $\mathcal{S}$. We call $\mathcal{S}$ the *model* of the L constraint equations. Optimally modifying an input figure so that the constraint equations are satisfied is equivalent to *orthogonally projecting* $\bigoplus_{\alpha=1}^N \boldsymbol{x}_\alpha$ to $\bigoplus_{\alpha=1}^N \hat{\boldsymbol{x}}_\alpha$ on the model $\mathcal{S}$ [2], [3] (Fig.3).

Consider the *residual*

$$\hat{J}=\sum_{\alpha=1}^N \|\boldsymbol{x}_\alpha-\hat{\boldsymbol{x}}_\alpha\|^2. \tag{1}$$

If $\boldsymbol{x}_1$, ..., $\boldsymbol{x}_N$ are perturbed from their true positions $\bar{\boldsymbol{x}}_1$, ..., $\bar{\boldsymbol{x}}_N$ by independent Gaussian noise of mean 0 and variance σ^2, it can be shown that $\hat{J}/\sigma^2$ is subject to a χ^2 distribution with L degrees in the first order [3]. Hence,

$$\hat{\sigma}^2=\frac{\hat{J}}{L} \tag{2}$$

is an unbiased estimator of the variance σ^2.

If the noise in the true positions $\bar{\boldsymbol{x}}_1$, ..., $\bar{\boldsymbol{x}}_N$ were different, we would observe different positions $\boldsymbol{x}_1^*$, ..., $\boldsymbol{x}_N^*$. Consider the following *expected residual*:

$$\hat{J}^*=\sum_{\alpha=1}^N \|\boldsymbol{x}_\alpha^*-\hat{\boldsymbol{x}}_\alpha\|^2. \tag{3}$$

It can be shown that *$\hat{J}$ is smaller than $\hat{J}^*$ by $2(2N-L)\sigma^2$ in expectation.* So, we define the *geometric AIC* as follows [2], [3]:

$$AIC(\mathcal{S})=\hat{J}+2(2N-L)\sigma^2. \tag{4}$$

Suppose the L' constraint equations of model $\mathcal{S}'$ can be obtained by adding some equations to the L constraint equations of model $\mathcal{S}$. Then, we say model $\mathcal{S}'$ is *stronger* than model $\mathcal{S}$, or model $\mathcal{S}$ is *weaker* than model $\mathcal{S}'$, and write

$$\mathcal{S}'\succ\mathcal{S}. \tag{5}$$

If the weaker model $\mathcal{S}$ holds, the variance σ^2 of the noise can be estimated by eq. (2) whether or not the stronger model $\mathcal{S}'$ is satisfied. Let $\hat{J}$ and $\hat{J}'$ be the corresponding residuals. The criterion based on the geometric AIC can be written as follows:

$$\frac{\hat{J}'}{\hat{J}}<\frac{2L'-L}{L}. \tag{6}$$

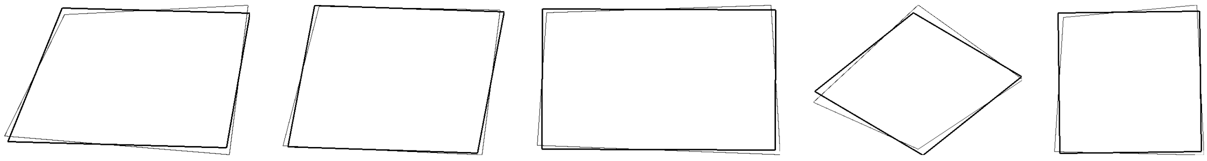

Fig. 4 Input figures and inferred shapes: (a) trapezoid; (b) parallelogram; (c) rectangle; (d) diamond; (e) square.

4. Classification Procedure

We turn back to quadrilaterals. Let $\hat{J}_{\rm trap}$ be the residual for optimally correcting $\boldsymbol{x}_1$, $\boldsymbol{x}_2$, $\boldsymbol{x}_3$, and $\boldsymbol{x}_4$ into vertices of a trapezoid. Similarly, let $\hat{J}_{\rm para}$, $\hat{J}_{\rm rect}$, $\hat{J}_{\rm diam}$, and $\hat{J}_{\rm squa}$ be the residuals for correcting the input figure into a parallelogram, a rectangle, a diamond, and a square, respectively. The procedure for discrimination is as follows:

1. a. If $\hat{J}_{\rm para} \leqq 3\hat{J}_{\rm trap}$, judge the input figure to be a parallelogram and go to Step 2.
   b. Output an optimally corrected trapezoid, and stop.
2. a. If $\hat{J}_{\rm rectangle} \leqq 2\hat{J}_{\rm para}$, judge the input figure to be a rectangle and go to Step 3.
   b. If $\hat{J}_{\rm diam} \leqq 2\hat{J}_{\rm para}$, judge the input figure to be a diamond and go to Step 4.
   c. Output an optimally corrected parallelogram, and stop.
3. a. If $\hat{J}_{\rm squa} \leqq (5/3)\hat{J}_{\rm rect}$, judge the input figure to be a square, output an optimally corrected square, and stop.
   b. Output an optimally corrected rectangle, and stop.
4. a. If $\hat{J}_{\rm squa} \leqq (5/3)\hat{J}_{\rm diam}$, judge the input figure to be a square, output an optimally corrected square, and stop.
   b. Output an optimally corrected diamond, and stop.

Note that *this procedure does not involve any empirical threshold* ϵ.

Figure 4 shows implementation examples: input figures are inferred to be (a) a trapezoid, (b) a parallelogram, (c) a rectangle, (d) a diamond, and (e) a square. The input figure is drawn in thin lines; the corrected figure is drawn in thick lines.

5. Concluding Remarks

We have presented a graphic interface for inferring regularity in a figure input by a mouse without using any empirical thresholds; it automatically imposes the inferred regularity. We argued that, unlike the usual pattern recognition problem, we cannot select the closest class measured by some distance criterion if the classes have *inclusion relations*. We then showed that this difficulty can be resolved by introducing the *geometric AIC*.

Although recognition and classification of features that have inclusion relations have been studied in relation to a variety of applications [5], [6], the intrinsic difficulty caused by the inclusion relations does not seem to be fully understood yet [4]. The approach presented here is expected to play an important role in dealing with such problems.

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