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A MICROPOLAR CONTINUUM THEORY FOR THE FLOW OF GRANULAR MATERIALS

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Abstract—The flow of granular materials at high deformation rates and low stress levels is described here using a micropolar continuum theory. The constitutive equations are obtained by statistical inference; the interparticle interactions are first studied, and constitutive relations of the continuum theory are deduced by statistical averaging. The material is assumed to consist of cohesionless rigid spheres of uniform size and the same mass, and the equivalence of the continuum description and the particle description is discussed. Particle fluctuations are regarded as macroscopic "heat", and a thermodynamic analogy is developed. The "equations of state" are derived by assuming "local equilibrium". "Entropy" is also introduced, and the law of "entropy" increase is proved. Two different regimes of flows—slow flows and fast flows—are analyzed, and such typical non-linear characteristics as the existence of the angle of repose and the "thermal dilatation" due to the particle fluctuations are shown to exist.

1. INTRODUCTION

A GRANULAR material is an aggregate of a large number of particles, and its mechanical properties have been studied in soil mechanics and powder mechanics. In soil mechanics, the quasi-static equilibrium at high stress levels has been analyzed on the assumption that the material is in the limiting equilibrium in accordance with the Morhr–Coulomb yield criterion[1–4]. Powder mechanics, on the other hand, has chiefly dealt with conveyance of granular materials by various equipments—hoppers, chutes, pipes, channels, conveyors, etc.[5, 6]. However, the understanding of the mechanics of granular materials is still insufficient due to the internal complexity of the material. Goodman and Cowin[7–10] took the solid volume fraction of the material as an internal variable and developed a general continuum theory. Their theory is intended to describe the flow of granular materials from the limiting equilibrium into flow regimes. However, their argument is restricted to formal continuum mechanical considerations, so that the material constants appearing in the theory are left undetermined and their constitutive assumptions cannot be interpreted in terms of the microscopic properties of the constituent granules.

There are two approaches to the modeling of the mechanical behaviors of granular materials: the *macroscopic* or *continuum* approach and the *microscopic* or *particulate* approach. In the particulate approach, one considers an ensemble of particles of finite size (idealized, say, as rigid spheres) and attempts to deduce the laws governing the mechanical behavior of the entire ensemble. Qualitative insight into the mechanical behaviors of packed sphere particles has been obtained with this approach[11, 12]. However, this approach is generally not adapted to the determination of quantitative results, because the results are very sensitive to the configuration of the packed particles. The continuum approach, on the other hand, is easily adapted to the derivation of quantitative equations. Most design criteria concerning granular materials are based on continuum concepts such as stress and strain. However, in continuum models, including the Goodman–Cowin model, the notion of discrete granules is no longer retained and the constitutive equations are determined either by experiments or by plausible assumptions (e.g. tensor polynomial expansions).

In this paper, we take a third approach, i.e. the mixed approach, to obtain quantitative continuum equations based on microscopic properties of the constituent particles. We take advantage of the fact that the particles are irregularly fluctuating when they are in motion. We take statistical averages of quantities involved in the interparticle interactions to obtain quantitative results insensitive to the particle configuration.

The theory presented here is intended to represent the flow of granular materials at relatively low stress levels and high deformation rates such that the elastic properties of the particles are negligible and all the particles are moving relative to each other, or, in other words, the bulk behavior is governed by interparticle friction and collisions alone. Hence, the theory is not intended to represent the stresses when the rate of deformation becomes zero. The effects of the fluid contained in the interstices and pneumatic effects are assumed to be negligible, which implies that the bulk solid consists of relatively large particles of fairly uniform size.

Furthermore, we assume that the volume concentration is close to that of packed particles, so that the mean free path of the particles is short compared to the size of the particles. This implies that the material is in the *grain-inertia region* of Bagnold[13, 14].

As a particulate model representing these assumptions, we consider flows of dry cohesionless rigid spheres of uniform size and of the same mass. The link connecting a particulate model with an equivalent continuum model is the conservation laws of mass, momentum, angular momentum and energy, in which kinematic quantities such as the velocity are particulate concepts and the stresses are continuum concepts. We regard the velocity and the rotation velocity of particles as two independent kinematic field variables, so that basic kinematic equations are given by the theory of polar or micropolar continua[15–20].

Experimental investigations have shown that there are, roughly speaking, two distinct flow regimes of the inclined gravity flows (Takahasi[21], Hayashi *et al.*[22], Oyama[23], Shoji[24]). One is a relatively slow incipient flow in which the particles are rolling over the lower layer of particles in an ordered manner. The other is a rapid developed flows in which all the particles are following chaotic paths and interacting vigorously with their neighbors. We shall first consider the former regime and show the perfectly plastic nature of the flow including the existence of the angle of repose.

We shall then consider the latter case and develop a thermodynamic analogy. Following Oshima and Ogawa[25, 26], we regard the internal particle fluctuation as macroscopic "heat". We introduce the concept of "local equilibrium" and deduce "equations of state". "Entropy" is also defined, and the law of "entropy" increase is formulated. As an illustrative example, the "thermal dilatation" of the flow due to the particle fluctuations is analyzed. This is a phenomena demonstrating the *normal stress effects* of granular materials observed by Bagnold[13, 14] and Savage[27].

Bagnold[13, 14] analyzed the particle oscillation in a shear flow with a scheme of rigid wall reflection and made experiments on both fluid suspended particles and dry cohesionless particles. We shall consider a 3-dimensional version of his scheme and obtain a consistent picture of the flow in "local equilibrium". We shall show that our model leads to "equations of state" which are in accordance with his observation including the power dependence of the stresses on the velocity gradient. Savage[27] made extensive experiments on the flow of cohesionless particles including such a singular phenomena as the surging and "hydraulic" jumps. His analysis is based on the constitutive assumptions of the Goodman-Cowin theory. The main feature of the Goodman-Cowin theory is the assumption that the gradient of the solid volume fraction influences the stresses. The contribution of the strain-rate to the stresses was first assumed to be linear, which implies that the material reduces to the Navier–Stokes fluid if the influence of the solid volume fraction is neglected. Later, the quadratic terms of the strain-rate were introduced in accordance with Bagnold's experiment. The present theory, on the other hand, is completely different from the Goodman-Cowin theory in the sense that our theory is based on the microscopic considerations of particle rotation, incorporating nonlinearity from the beginning.

2. THEORY FOR SLOW FLOWS

2.1 *Conservation laws of micropolar continua*

We first consider relatively slow and fairly ordered flows such that the bulk behavior is governed mainly by interparticle friction. In our continuum description, the velocity v_i and the rotation ω_{ji} of the particles are represented by continuous functions of position and time. Here, ω_{ji} is a skew symmetric tensor expressing the angular velocity about an axis perpendicular to the $[ji]$ coordinate plane. We interprete the functions $v_i(x, t)$ and $\omega_{ji}(x, t)$ as the averaged values over a small region about the coordinate point x at time t. The region is assumed to contain several particles and to be small compared to the macroscopic scale of the flow pattern. We call the region the *macroelement* after Eringen and Suhubi[19]. The passage from a discrete model to an equivalent continuum model is discussed in many papers (e.g. Kröner[28]).

In the following, we adopt tensor notation and the summation convention for repeated indices. The partial derivation $\partial/\partial x^i$ is denoted by ∂_i. The symmetrization and the alternation of tensor indices are indicated by ( ) and [ ], respectively, i.e.

$$A_{(ji)} = \frac{1}{2}(A_{ji} + A_{ij}), \quad A_{[ji]} = \frac{1}{2}(A_{ji} - A_{ij}).$$

The coordinate system is always Cartesian, so that we do not make any distinction between contravariant and covariant components of tensors.

Now, we take the velocity v_i and the rotation ω_{ji} (also referred to as *spin* or *gyration*) to be independent kinematic field variables, so that the model continuum is a polar or micropolar continum[15–20]. The conservation laws of mass, momentum and angular momentum are respectively expressed in the following integral forms

$$\frac{d}{dt}\int_V \rho \, dV = 0, \tag{1}$$

$$\frac{d}{dt}\int_V \rho v^i \, dV = \int_V \rho b^i \, dV + \int_S \sigma^{ji} n_j \, dS, \tag{2}$$

$$\frac{d}{dt}\int_V \left[2\rho x^{[j} v^{i]} + \frac{2}{5}\rho a^2 \omega^{ji}\right] dV = \int_V 2\rho x^{[j} b^{i]} \, dV + \int_S [2x^{[j}\sigma^{|k|i]} n_k + \mu^{kji} n_k] \, dS. \tag{3}$$

Here, the domain of integration V is any part of the material moving in the flow but is large compared to the macroelement dV. The vector n_i is the unit normal to the boundary surface S surrounding the region V, and a is the radius of the sphere particles which is assumed to be small compared to the size of the region V. By | | we mean that the alternation is not applied to the indices in | |, and x^i is the position vector. The bulk density ρ and the body force per unit mass, b^i, are continuous quantities obtained by taking average over the macroelement, while the stress σ^{ji} and the couple-stress μ^{kji} are continuum-mechanical concepts. We regard (2) and (3) as the defining equations of these stresses, i.e. they are the averaged surface interactions over the surface element dS, small but finite, such that (2) and (3) hold for any region V in the previously stated sense. (The detailed discussion of this problem may be found in [28]).

Application of Gauss' theorem transforms (1)–(3) into the following macroscopic, or continuum-mechanical, kinematic equations

$$\frac{d\rho}{dt} + \rho \partial_i v^i = 0, \tag{4}$$

$$\rho \frac{dv^i}{dt} = \partial_j \sigma^{ji} + \rho b^i, \tag{5}$$

$$\frac{2}{5}\rho a^2 \frac{d\omega^{ji}}{dt} = \partial_k \mu^{kji} + 2\sigma^{[ji]}. \tag{6}$$

where $d/dt = \partial/\partial t + v^i \partial_i$. Note that the transformation is achieved using the fact that, in continuum description, every quantity becomes uniform when the size of the material region V approaches the size of the macroelement. In the slow flow regime, we assume that the material is uniform and incompressible, i.e. $\partial_i v^i = 0$ and $\rho = \text{const}$. Hence, (4) is identically satisfied.

2.2 *Energy dissipation and equivalent stresses*

Now, we consider the conservation law of energy. Since there is no potential force acting between particles and the particles are rigid, we need not consider potential energy. Let $d'W/dt$ be the rate of work done by external forces to the material region V and let K be the kinetic energy of the material in V. They are, in continuum description, expressed, respectively, by

$$\frac{d'W}{dt} = \int_V \rho b^i v_i \, dV + \int_S \left[\sigma^{ji} n_j v_i + \frac{1}{2}\mu^{kji} n_k \omega_{ji}\right] dS, \tag{7}$$

$$K = \int_V \left[\frac{1}{2}\rho v_i v^i + \frac{1}{10}\rho a^2 \omega_{ji}\omega^{ji}\right] dV. \tag{8}$$

The conservation law of energy is written as

$$\frac{\mathrm{d}'W}{\mathrm{d}t}-\frac{\mathrm{d}K}{\mathrm{d}t}=\int_V \Phi\,\mathrm{d}V, \tag{9}$$

where Φ is the rate of energy dissipation due to the interparticle friction in a unit volume. Applying Gauss' theorem and taking account of the incompressibility, we obtain the macroscopic relation

$$\Phi=\bar{\sigma}^{ji}E_{ji}+\sigma^{[ji]}R_{ji}+\frac{1}{2}\mu^{kji}\Omega_{kji}, \tag{10}$$

where

$$\bar{\sigma}^{ji}\equiv\sigma^{(ji)}-\frac{1}{3}\delta_{ji}\sigma^{kk},\quad E_{ji}\equiv\partial_{(j}v_{i)}-\frac{1}{3}\delta_{ji}\partial_k v^k,$$
$$R_{ji}\equiv\partial_{[j}v_{i]}-\omega_{ji},\quad \Omega_{kji}\equiv\partial_k\omega_{ji} \tag{11}$$

and δ_{ji} is the Kronecker delta.

We can see that E_{ji}, R_{ji} and Ω_{kji} are the basic quantities representing the deformation rates of the material. In the linearized theories of polar or micropolar continua, the dissipation Φ is assumed to be quadratic in E_{ji}, R_{ji} and Ω_{kji}. Then, the stresses are given by

$$\bar{\sigma}^{ji}=\frac{1}{2}\frac{\partial\Phi}{\partial E_{ji}},\quad \sigma^{[ji]}=\frac{1}{2}\frac{\partial\Phi}{\partial R_{ji}},\quad \frac{1}{2}\mu^{kji}=\frac{1}{2}\frac{\partial\Phi}{\partial\Omega_{kji}}. \tag{12}$$

However, we do not make any constitutive assumption on the form Φ. Instead, we deduce the form of Φ by microscopic particulate consideration. We first analyze microscopic energy dissipation due to the interparticle friction and convert the results to macroscopic, or continuous relations by statistical inference. Then, we determine the stresses in such a way that (10) is identically satisfied. Suppose we have determined the form $\Phi(E_{ji}, R_{ji}, \Omega_{kji})$. In most problems of mechanics, we find that, if some particular mechanism of energy dissipation (e.g. friction, viscosity, etc.) is given, the form of Φ becomes homogeneous in its arguments. Let k be the degree of homogeneity of Φ. Then, according to Euler's theorem

$$\Phi=\frac{1}{k}\left(\frac{\partial\Phi}{\partial E_{ji}}E_{ji}+\frac{\partial\Phi}{\partial R_{ji}}R_{ji}+\frac{\partial\Phi}{\partial\Omega_{kji}}\Omega_{kji}\right). \tag{13}$$

Our choice here is

$$\bar{\sigma}^{ji}=\frac{1}{k}\frac{\partial\Phi}{\partial E_{ji}},\quad \sigma^{[ji]}=\frac{1}{k}\frac{\partial\Phi}{\partial R_{ji}},\quad \frac{1}{2}\mu^{kji}=\frac{1}{k}\frac{\partial\Phi}{\partial\Omega_{kji}}, \tag{14}$$

which is a non-linear or non-Newtonian generalization of (12). Of course, there are other choices to satisfy (10), but we expect that the stresses so determined can well characterize the bulk behaviors of the flow, because they correctly reflect the energy dissipation in the material. This kind of reasoning is often employed in the analysis of non-linear mechanical systems[29].

The above procedure is closely related to the variational formulation of the mechanics of non-Newtonian fluids. As is well known, the inertia term $v^j\partial_j v^i$ of non-ideal fluids cannot generally be derived by the variational principle[30, 31]. Hence, as usual, we consider now only steady flows in which the inertia terms can be neglected. Then, we have:

Theorem 1. *Let V_0 be a fixed region and let the boundary conditions consist of specifying the values of v_i and ω_{ji} and assigning free surface. Then, under the assumption of* (14), *the possible flow is such that*

$$\Psi[v_i,\omega_{ji}]=\int_{V_0}\left(\frac{1}{k}\Phi-p\partial v^i-\rho b^i v_i\right)\mathrm{d}V \tag{15}$$

takes extremum, where $p = -(1/3)\delta_{ji}\sigma^{kk}$.

Proof. Let $v_i \to v_i + \delta v_i$ and $\omega_{ji} \to \omega_{ji} + \delta\omega_{ji}$. Then, the first variation of Ψ is

$$\delta\Psi = \int_{V_0}\left[\frac{1}{k}\left(\frac{\partial\Phi}{\partial E_{ji}}\delta E_{ji} + \frac{\partial\Phi}{\partial R_{ji}}\delta R_{ji} + \frac{\partial\Phi}{\partial\Omega_{kji}}\delta\Omega_{kji}\right) - p\partial_i\delta v^i - \rho b^i\delta v_i\right]\mathrm{d}V$$

$$= \int_{V_0}\left[\bar{\sigma}^{ji}\partial_j\delta v_i + \sigma^{[ji]}(\partial_j\delta v_i - \delta\omega_{ji}) + \frac{1}{2}\mu^{kji}\partial_k\delta\omega_{ji} - p\partial_i\delta v^i - \rho b^i\delta v_i\right]\mathrm{d}V$$

$$= -\int_{V_0}\left[\left(\partial_j\sigma^{ji} + \rho b^i\right)\delta v_i + \left(\frac{1}{2}\partial_k\mu^{kji} + \sigma^{[ji]}\right)\delta\omega_{ji}\right]\mathrm{d}V$$

which vanishes according to (5) and (6). Here, we have put the surface terms zero according to our assumptions on boundary conditions.

We can eliminate the term $-p\partial_i v^i$ in (15) by assigning the constraint of incompressibility. If the boundary conditions consist of specifying the values of v_i and ω_{ji} only and if the body force is a potential force, then we can also omit the term $\rho b^i v_i$ [31]. We will see later that the extremum is minimum in our theory. Thus, our principle turns out to be the so called *minimum dissipation principle* which is valid for wide range of non-Newtonian fluids (e.g. power law fluids).

2.3 *Interparticle friction and constitutive equations*

Now, we turn our attention to microscopic, or particulate consideration. Consider two rotating particles in contact with each other as shown in Fig. 1. We inbed this set of particles into the continuum description by putting

$$v_i^* = v_i + 2a\nu_j D_{ji}, \quad \omega_{ji}^* = \omega_{ji} + 2a\nu_k\Omega_{kji}, \tag{16}$$

in the sense of statistical average, where $D_{ji} \equiv \partial_j v_i$ and ν_i is a unit vector originating at the center of the particle and indicating the direction of contact (see Fig. 1). Here, D_{ji} and Ω_{kji} are the gradients of the velocity and the rotation and hence they are macroscopic quantities of continuum description. The velocities of the particles at the contact point are $v_i + a\nu_j\omega_{ji}$ and $v_i^* - a\nu_j\omega_{ji}^*$. Then, the tangential component of the relative velocity at the contact point is

$$\xi_i = 2a(\nu_j D_{ji} - \nu_j\omega_{ji} - \nu_k\nu_j\nu_i D_{kj} - a\nu_k\nu_j\Omega_{kji}). \tag{17}$$

Since the direction of ν_i is completely random in a flow, it is regarded as a random variable whose probability distribution is uniform over the whole solid angle 4π. Put the root mean square of ξ_i to be ξ, i.e.

$$\xi \equiv \sqrt{(\overline{\xi_i\xi_i})}, \tag{18}$$

where the bar denotes the average with respect to ν_i over the whole solid angle. Making use of the identities

$$\overline{\nu_j\nu_i} = \frac{1}{3}\delta_{ji}, \quad \overline{\nu_k\nu_j\nu_i} = 0, \quad \overline{\nu_m\nu_l\nu_k\nu_j\nu_i} = 0,$$
$$\overline{\nu_l\nu_k\nu_j\nu_i} = \frac{1}{15}(\delta_{lk}\delta_{ji} + \delta_{lj}\delta_{ki} + \delta_{li}\delta_{kj}), \tag{19}$$

we obtain

$$\xi = \frac{2\sqrt{6}}{3}a\hat{\omega}, \tag{20}$$

where

$$\hat{\omega} \equiv \sqrt{\frac{3}{10}E_{ji}E_{ji} + \frac{1}{2}R_{ji}R_{ji} + \frac{a^2}{10}(\Omega_{kkj}\Omega_{llj} + \Omega_{kji}\Omega_{kji} + \Omega_{kji}\Omega_{jki})}. \tag{21}$$

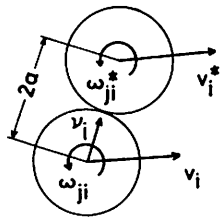

Fig. 1.

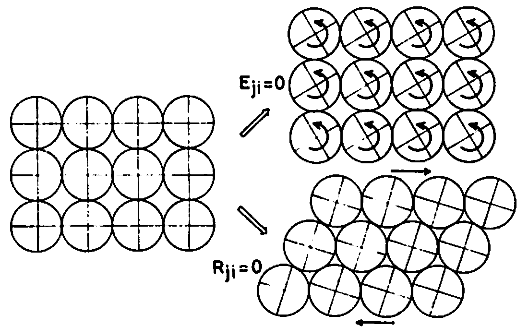

Fig. 2.

Fig. 1. Rotating particles in contact with each other.

Fig. 2. Minimum amount of interparticle friction in a shear motion.

Let us consider the physical meaning of $\hat{\omega}$, which is a macroscopic quantity. Suppose the positions of the particles are fixed (i.e. $v_i = 0$) and the rotation is uniform (i.e. $\partial_k\omega_{ji} = 0$). Then

$$\hat{\omega} = \sqrt{\frac{1}{2}\,\omega_{ji}\omega_{ji}}\,, \tag{22}$$

which is the magnitude of the angular velocity. Hence, we can regard $\hat{\omega}$ as the magnitude of particle rotation relative to the velocity field. On the other hand, if the particle rotation is *constrained* to the velocity field (i.e. $\omega_{ji} - \partial_{[j}v_{i]} = 0$ or $R_{ji} = 0$) and the rotation is uniform (i.e. $\partial_k\omega_{ji} = 0$), then

$$\hat{\omega} = \sqrt{\frac{3}{10}\,E_{ji}E_{ji}}\,, \tag{23}$$

which represents the minimum amount of friction in a shear flow schematically shown in Fig. 2. Thus, $\hat{\omega}$ represents the total amount of interparticle friction with both the shearing and the particle rotation taken into account.

Now, we consider the contact forces of particles. Take a particular particle and let f_K ($K = 1, 2, \ldots, N$) be the normal component of the contact force at the K-th contact point, where N is the number of the contact points of the particle (see Fig. 3). Put $\bar{f} = \Sigma f_K/N$. In order to determine the magnitude of $\bar{f}$, we consider virtual compression by the ratio γ. The virtual work done by f_K's is $\Sigma f_K/4\pi a^2 \times \gamma \times (4/3)\pi a^3 = (1/3)aN\bar{f}\gamma$ per a single particle. Multiplying it by ρ/m, the number of particles in a unit volume, we find that the virtual work done in a unit volume is $(1/3)N\bar{f}(\rho/m)a\gamma$. This work must be equal to the virtual work $p\gamma$ done by the macroscopic pressure p, because p is the Lagrange multiplier with respect to the constraint of incompressibility (see (15)). Hence, we have

$$\bar{f} = \frac{3mp}{N\rho a}\,. \tag{24}$$

The average rate of energy dissipation at the K-th contact point is approximated by $\mu f_K\xi$,

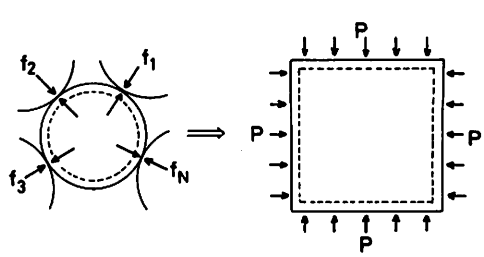

Fig. 3. The contact forces and the equivalent macroscopic pressure.

where μ is the kinetic friction coefficient. Then, the rate of energy dissipation for a single particle is $N\mu\bar{f}\xi$. Multiplying it by ρ/m, the number density, and dividing it by 2, because each contact is doubly counted when summed, we finally obtain the macroscopic energy dissipation rate Φ per unit volume. By the use of (20) and (24), the final form of Φ becomes

$$\Phi = \sqrt{6}\mu p\hat{\omega}(E_{ji}, R_{ji}, \Omega_{kji}). \tag{25}$$

Applying the procedure discussed in the preceding section, we obtain the following constitutive equations.

$$\sigma^{ji} = -p\delta_{ji} + \bar{\sigma}^{ji} + \sigma^{[ji]}, \tag{26}$$

$$\bar{\sigma}^{ji} = \frac{3\sqrt{6}\mu}{10}\frac{p}{\hat{\omega}}\left(\partial_{(j}v_{i)} - \frac{1}{3}\delta_{ji}\partial_k v^k\right), \tag{27}$$

$$\sigma^{[ji]} = \frac{\sqrt{6}\mu}{2}\frac{p}{\hat{\omega}}(\partial_{[j}v_{i]} - \omega_{ji}), \tag{28}$$

$$\mu^{kji} = \frac{\sqrt{6}\mu a^2}{5}\frac{p}{\hat{\omega}}(\delta_{k[j}\partial_{|l}\omega_{l|i]} + \partial_k\omega_{ji} - \partial_{[j}\omega_{i]k}). \tag{29}$$

The stresses have homogeneous forms of degree 0 in $\partial_j v_i$, ω_{ji} and $\partial_k\omega_{ji}$. Hence, there is no one-to-one correspondence between the velocity field (including the rotation) and the stresses. In this sense, we can say that the flow is a *perfect plastic flow*[32]. This is one of the consequences of the fact that the stresses are caused by interparticle friction, whose magnitude does not depend on the friction velocity.

2.4 *Angle of repose of inclined gravity flows*

Consider the gravity flow on an inclined plate as shown in Fig. 4. The velocity is assumed to be parallel to the x-axis, and all the quantities are assumed to be functions of y alone. The equation of motion for the y-component of the velocity is easily integrated to yield

$$p = \rho g(h - y)\cos\theta, \tag{30}$$

where g is the gravitational acceleration, h is the depth of the flow, and θ is the angle of inclination.

If $a/h \ll 1$, the term $(a^2/10)(\cdots)$ in (21) becomes small compared to the preceding terms. If that term is neglected, then the couple-stress μ^{kji} vanishes. Then, we can see from (6) and (28) that the particle rotation tends to the *constrained rotation*, i.e. $R_{ji} = 0$. If the particle rotation is the constrained rotation and if $\partial u/\partial y > 0$, the shear stress σ^{yx} becomes

$$\sigma^{yx} = p\tan\theta^* \tag{31}$$

where

$$\tan\theta^* = \frac{3\sqrt{10}}{10}\mu. \tag{32}$$

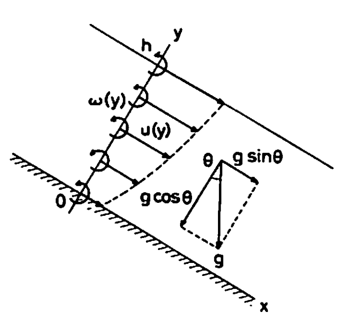

Fig. 4. Inclined gravity flow.

We can call θ^* the *angle or repose* of the material. The equation of motion for u becomes in this case

$$\frac{\partial u}{\partial t} = g \cos \theta(\tan \theta - \tan \theta^*). \tag{33}$$

Thus, we can see that, if $\theta > \theta^*$, the flow is accelerated, while, if $\theta < \theta^*$, the flow is damped. When $\theta = \theta^*$, the flow pattern is indeterminate. If $\mu = 0.7$ for glass, then $\theta^* = 33.6°$, which seems to be an adequate value for the angle of repose[5].

If we assign the boundary condition that $u = 0$ and $\omega = 0$ at $y = 0$, then the couple-stress arises and the particle rotation is not completely constrained to the velocity. However, it can easily be shown that the influence of the couple-stress is restricted to the area near the boundary, the width of which is about a few times the size of the particles. This short range property of the couple-stress has been discussed by several authors in the theory of the couple-stress (e.g. [16, 28]).

3. THEORY FOR FAST FLOWS

3.1 *Particle fluctuation and a thermodynamic analogy*

We now consider rapid developed flows in which all the particles are irregularly fluctuating. Let $v_i(P)$ and $\omega_{ji}(P)$ be the velocity and the angular velocity, respectively, of particle at the point P. They are partitioned into two parts such as

$$v_i(P) = v_i + v_i'(P), \quad \omega_{ji}(P) = \omega_{ji} + \omega_{ji}'(P), \tag{34}$$

where v_i and ω_{ji} are the averaged values over the macroelement around the point P, while $v_i'(P)$ and $\omega_{ji}'(P)$ are quantities regarded as random variables of zero mean representing the irregular fluctuations due to the interparticle collisions. The kinetic energy averaged over the macroelement is

$$\frac{1}{2}\rho v_i v_i + \frac{1}{10}\rho a^2 \omega_{ji}\omega_{ji} + \frac{1}{2}\rho \overline{v_i' v_i'} + \frac{1}{10}\rho a^2 \overline{\omega_{ji}'\omega_{ji}'}, \tag{35}$$

where the bar means the average over the macroelement. The first two terms of (35) are the kinetic energy density of the material in the continuum description. The remaining terms must be regarded as "internal energy" of the material in the continuum description. Being an averaged quantity, the "internal energy" is a macroscopic continuous quantity. We put the "internal energy" per unit mass to be

$$\epsilon = \frac{1}{2}\overline{v_i' v_i'} + \frac{1}{10} a^2 \overline{\omega_{ji}'\omega_{ji}'}. \tag{36}$$

The existence of the "internal energy" does not affect the conservation laws (1)–(3), for the integrands in the left-hand sides are linear forms in v_i and ω_{ji}. Hence, we have (4)–(6) also for developed flows.

Consider the energy conservation law. Part of the kinetic energy of the particles is lost in the flow by the interparticle friction and collisions. We look on the energy loss as negative "heat" supply, where by "heat" we mean the irregular fluctuations of the particles. In the following, we adopt notions of thermodynamics and develop a thermodynamic analogy, but we use " " to emphasize our usage distinct from the usual ones. The conservation law of energy is written as

$$\frac{dK}{dt} + \frac{dU}{dt} = \frac{d'W}{dt} + \frac{d'Q}{dt}, \tag{37}$$

where $d'W/dt$ and K are given by (7) and (8), respectively, and

$$U = \int \rho\epsilon \, dV, \tag{38}$$

$$\frac{d'Q}{dt} = \int q \, dV - \int h^i n_i \, dS. \tag{39}$$

Here, q is the "heat" supply per unit volume, i.e. $-q$ is the rate of energy dissipation in a unit volume, while h^i is the "heat flux", i.e. the transmission of the fluctuation energy to neighboring particles. Substitution of (7), (8), (38) and (39) in (37) with application of Gauss' theorem yields

$$\rho \frac{d\epsilon}{dt} = -p\partial_i v^i + \Phi + q - \partial_i h^i, \tag{40}$$

where $p = -(1/3)\delta_{ji}\sigma^{kk}$, and Φ is the form defined by (10).

If we can successfully give constitutive equations for σ^{ji}, μ^{kji}, q and h^i in terms of ρ, v_i, ω_{ji} and ϵ, then eqns (4)–(6) and (40) give the evolution equations for ρ, v_i, ω_{ji} and ϵ, respectively. However, this process is overdescription in a sense. Suppose, for example, the material is macroscopically at rest (i.e. $v_i = 0$, $\omega_{ji} = 0$). Even in this case, we cannot conclude that all the particles are at rest, because the particle fluctuations may still exist depending upon the history of the past motion. However, we can expect that such fluctuations are soon damped. This consideration suggests the concept of "equilibrium". Even if the material is in motion, we can expect that the particle fluctuations soon grows up to an extent determined by the macroscopic flow pattern. We say that a flow is in "local equilibrium", if the "internal energy" ϵ depends not on the history of motion but only of the present state of the flow in the form $\epsilon = \epsilon(v_i, \partial_j v_i, \omega_{ji}, \partial_k\omega_{ji})$, which is called the "equation of state" for "internal energy". Since we are, in many practical applications, not interested in minute transient processes, it is this "local equilibrium" that is of our concern.

In a "local equilibrium" flow, the work done by the stresses are classified into two categories, one is the "thermal work" done by the pressure p caused by the interparticle collisions. This work directly changes the "internal energy" of the material. The other is the "dissipative work" spent in the interparticle friction and collisions. We assume that the bulk viscosity can be neglected, which implies that we are considering only those flows which do not change their volume so rapidly. We further assume that, in "local equilibrium", the fluctuations of a particle is nearly the same as those of surrounding ones, so that the "heat flux" can be neglected in "local equilibrium". Equation (40) is then decomposed into two parts

$$\rho \frac{d\epsilon}{dt} = -p\partial_i v^i, \qquad -q = \Phi. \tag{41}$$

3.2 *Equations of state and constitutive equations*

Now, we consider particle models in order to deduce "equations of state". As has been already stated, we are considering those flows whose bulk density ρ is close to ρ_0, the bulk density of randomly packed spheres. This implies that the mean free path of the particles is short compared to the size of the particles. Hence, each particle in the flow is almost always repeating collisions to the nearest particles. In order to model such collisions, we adopt an artifice which is a 3-dimensional version of Bagnold's rigid wall reflection model [13, 14].

Consider a macroelement in the flow. The volume of the macroelement divided by the number of particles in it is $V = m/\rho$, which is the volume assigned to one particle. Let us call it the *occupation volume* of the particles. Define the *occupation radius* $r = (3V/4\pi)^{1/3}$ as the radius of a sphere whose volume equals the occupation volume. Now, a particle is assumed to be repeating collisions against a rigid spherical wall of radius r, the occupation radius, with fluctuation velocity v' (see Fig. 5). With this simple artifice, we can avoid detailed investigation of many body collisions of the particles. The mechanism that sustains the "local equilibrium" is considered later.

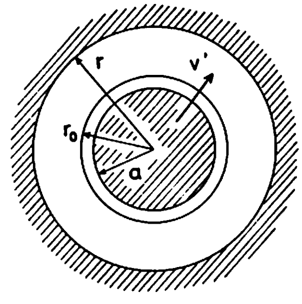

Fig. 5. The scheme of wall approximation.

Let r_0 be the occupation radius of randomly packed spheres. The particle travels without collisions over the distance $2(r-r_0)$, which is the mean free path of the particle. Hence, it collides against the wall $v'/2(r-r_0)$ times per unit time. The momentum given to the wall is $2mv'$ for each collision. Thus, the pressure on the wall is $p = mv'^2/4\pi r^2(r-r_0) = \rho\rho_0 v'^2/3(\rho_0 - \rho^{1/3}\rho_0^{2/3})$. Since we are considering the case $\rho \simeq \rho_0$, we can expand the denominator; $\rho_0 - \rho^{1/3}\rho_0^{2/3} = (\rho_0-\rho)/3 + (\rho_0-\rho)^2/9\rho + \cdots$. Retaining only the first term, we obtain

$$p = \frac{\rho_0 \rho v'^2}{\rho_0 - \rho}. \tag{42}$$

Next, consider the energy dissipation rate Φ spent in the interparticle friction. We assume, for simplicity's sake, that contribution of the fluctuation term ω'_{ji} can be neglected in comparison to the main term ω_{ji}. Then, the argument in Section 2.3 can be applied again. We finally have

$$\Phi = \sqrt{6}\mu\left(\frac{a}{r}\right)p\hat{\omega}. \tag{43}$$

Finally, we determine the "equation of state" for the "internal energy", considering the mechanism that sustains the "local equilibrium". The form $\epsilon(v_i, \partial_j v_i, \omega_{ji}, \partial_k\omega_{ji})$ must be invariant to translations and rigid rotations of the coordinate system, which is referred to as the *principle of objectivity* [33] or the *principle of material-frame indifference* [34]. Hence, we can conclude that ϵ is a function of E_{ji}, R_{ji} and Ω_{kji}. Since $\hat{\omega}(E_{ji}, R_{ji}, \Omega_{kji})$ is the quantity representing the interparticle interactions, we can naturally expect that ϵ is a function of $\hat{\omega}$. Once the form of ϵ is assumed, we have only to consider the special situation in which the average velocity of the particles is zero. In this situation, the kinetic energy of rotation is $(1/5)ma^2\hat{\omega}^2$, becaust $\hat{\omega}$ coincides with the magnitude of the particle rotation in this situation (see (22)). Since the second term in the right-hand side of (36) has the factor of a^2 and is small compared with the first term, we neglect the fluctuations of ω_{ji}. Then, the kinetic energy of the fluctuation mode is $(1/2)mv'^2$ in accordance with the scheme of wall approximation. We can expect that the latter kinetic energy increases as the former increases in "local equilibrium". We now postulate that the total kinetic energy of the particles is partitioned into these two modes of motion in a fixed ratio in the "local equilibrium". Thus, $(1/2)mv'^2 = T_\epsilon(1/5)ma^2\hat{\omega}^2$, where T_ϵ is the proportionality constant. Hence, we have

$$\epsilon = \frac{1}{5}T_\epsilon \rho a^2 \hat{\omega}^2, \tag{44}$$

which is the required "equation of state" for "internal energy". If we can further assume the *equipartition principle* of statistical mechanics, we have $T_\epsilon = 1$, but it seems difficult to provide physical justification for it. Therefore, we left the value of T_ϵ undetermined. The constant T_ϵ is possibly a function of the coefficient of restitution and the kinetic friction coefficient. Now we have completed the description of "local equilibrium". The mainstream supplies energy to the particle fluctuations and sustains the "local equilibrium". The supply rate depends on the interparticle reaction $\hat{\omega}(E_{ji}, R_{ji}, \Omega_{kji})$, the friction coefficient and the coefficient of restitution.

From (44), the fluctuation velocity v' is determined in the form

$$v' = \frac{\sqrt{10}}{5}\sqrt{T_e}\, a\hat{\omega}. \tag{45}$$

Substitution of this in (42) and (43) yields the following "equations of state" for the pressure p and the energy dissipation Φ.

$$p = \frac{2}{5} T_e a^2 \frac{\rho_0 \rho^2}{\rho_0 - \rho} \hat{\omega}^2, \tag{46}$$

$$\Phi = \frac{2\sqrt{6}}{5} T_e \mu \frac{a^3}{r} \frac{\rho_0 \rho}{\rho_0 - \rho} \hat{\omega}^3. \tag{47}$$

The dissipative stresses are determined by the procedure discussed in Section 2.2 as follows

$$\bar{\sigma}^{ji} = \frac{3}{10} C(\rho)\hat{\omega}\left(\partial_{(j}v_{i)} - \frac{1}{3}\delta_{ji}\partial_k v^k\right), \tag{48}$$

$$\sigma^{[ji]} = \frac{1}{2} C(\rho)\hat{\omega}(\partial_{[j}v_{i]} - \omega_{ji}), \tag{49}$$

$$\mu^{kji} = \frac{1}{5} a^2 C(\rho)\hat{\omega}(\delta_{k[j}\partial_{|l}\omega_{l|i]} + \partial_k\omega_{ji} - \partial_{[j}\omega_{i]k}), \tag{50}$$

where

$$C(\rho) \equiv \frac{2\sqrt{6}}{5} T_e \mu \frac{a^3}{r} \frac{\rho_0 \rho}{\rho_0 - \rho}. \tag{51}$$

3.3 *"Thermal dilatation" of inclined gravity flows*

We can see from (48)–(51) that the pressure p and the dissipative stresses are inherently associated with the shear motion of the flow as was observed by Bagnold[13, 14] and Savage[27]. This means that the material exhibits the so called *normal stress effects.* In order to study this, we again consider the inclined gravity flow shown in Fig. 4. Consider the case $a/h \ll 1$, and neglect the term $(a^2/10)(\cdots)$ in $\hat{\omega}$ as in Section 2.4. If the flow is steady and the particle rotation is constrained to the velocity, the shear stress σ^{yx} and the normal stress p $(= -\sigma^{yy})$ are given by

$$\sigma^{yx} = \frac{3\sqrt{15}}{200} C(\rho)\left(\frac{du}{dy}\right)^2, \tag{52}$$

$$p = \frac{\sqrt{6}}{40\mu}\frac{r}{a} C(\rho)\left(\frac{du}{dy}\right)^2. \tag{53}$$

Both of them are proportional to $(du/dy)^2$ as was derived and experimentally confirmed by Bagnold[13]. If we put the ratio of the former to the latter to be $\tan\theta^*$, then we obtain

$$\tan\theta^* = \frac{3\sqrt{10}}{10}\mu\frac{a}{r}, \tag{54}$$

which is a modified form of (32). The angle θ^* may be called the *internal angle of friction.*

If the flow is the simple shear flow, the equation of motion for the y-component of the velocity is easily integrated to give the form of $\rho(y)$. The integration constant is determined by the mass conservation relation

$$\int_0^\infty \rho\, dy = \rho_0 h_0, \tag{55}$$

where h_0 is the depth of the slab when all the particles are at rest. The final result is expressed in the following implicit form

$$\frac{y}{h_0} = 1 + \alpha\left(1 - \log\frac{\alpha\rho}{\rho_0 - \rho}\right) - \frac{\rho_0}{\rho_0 - \rho}, \tag{56}$$

where

$$\alpha \equiv \frac{3T_e a^2}{50\, g h_0 \cos\theta}\left(\frac{\mathrm{d}u}{\mathrm{d}y}\right)^2. \tag{57}$$

Figure 6 shows the density profile for the flow. It is seen that the increase of the shearing leads to the increase of the "internal energy", which in turn causes the "thermal dilatation" of the flow. Figure 7 shows the mass flux profile. The maximum flux in the flow does not necessarily occur at the upper surface. Similar results were also obtained in [7].

3.4 "*Entropy*" *of non-equilibrium flows*

Let us briefly consider flows not in "local equilibrium", although this problem is for the most part left to future investigation. In non-equilibrium flows, the "internal energy" ϵ is an independent quantity depending upon the history of motion. However, we can consider the "equation of state" (44) to be still valid, regarding T_e, instead, not as a constant but as a new independent physical quantity T which depends on the history of motion. The "equation of state" (46) for the pressure p is rewritten in terms of the occupation volume V as

$$p(V - V_0) = \frac{2}{5} Tma^2\hat{\omega}^2, \tag{58}$$

where V_0 is the occupation volume of randomly packed spheres. By analogy with the equation of state for ideal gass, we can say that T plays the role of *temperature*. Hence, we call T the "temperature" of the flow. From this view-point, the interpretation of Fig. 6 is that the rise of "temperature" causes the "thermal dilatation".

Let $\mathrm{d}\epsilon/\mathrm{d}t$ be partitioned into two parts such as

$$\frac{\mathrm{d}\epsilon}{\mathrm{d}t} = \frac{\mathrm{d}^e\epsilon}{\mathrm{d}t} + \frac{\mathrm{d}^*\epsilon}{\mathrm{d}t}, \tag{59}$$

where

$$\rho\frac{\mathrm{d}^e\epsilon}{\mathrm{d}t} \equiv -p\partial_i v^i. \tag{60}$$

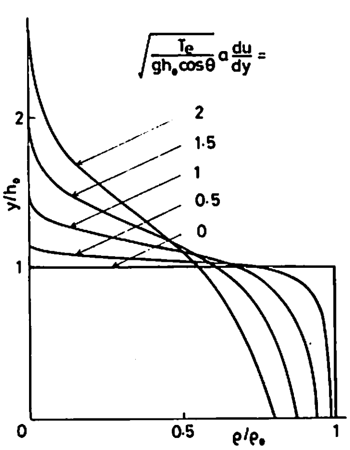

Fig. 6. Density profile for inclined gravity flow.

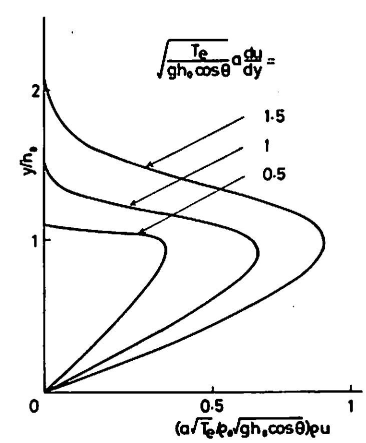

Fig. 7. Mass flux profile for inclined gravity flow.

From (40)

$$\rho \frac{d^*\epsilon}{dt} = \Phi + q - \partial_i h^i. \tag{61}$$

By definition, the non-equilibrium part $d^*\epsilon/dt$ vanishes in "local equilibrium". In order to assure the approach to "local equilibrium", we postulate that $d^*\epsilon/dt > 0$ for $T < T_e$ and that $d^*\epsilon/dt < 0$ for $T > T_e$. The "heat flux" h^i is assumed to be in the direction of lower "temperature". These assumptions are written as

$$\frac{1}{T - T_e}\frac{d^*\epsilon}{dt} < 0, \tag{62}$$

$$h^i \partial_i T < 0. \tag{63}$$

From (62) and (63), we can construct an "entropy" formulation. Let the "entropy" s per unit mass be defined as a quantity which obeys the following equation.

$$\rho \frac{ds}{dt} = -\frac{\Phi}{T - T_e}. \tag{64}$$

The initial value of s is irrelevant. Then, we have

Theorem 2. *Let the "entropy production rate" η be defined by the following "entropy balance equation".*

$$\frac{d}{dt}\int_V \rho s \, dV = \int_V \eta \, dV + \int_V \frac{q}{T - T_e} dV - \int_S \frac{h^i}{T - T_e} n_i \, dS. \tag{65}$$

Then

$$\eta > 0. \tag{66}$$

Proof. Applying Gauss's theorem to (65), we have

$$\eta = \rho \frac{ds}{dt} - \frac{q - \partial_i h^i}{T - T_e} - \frac{h^i \partial_i T}{(T - T_e)^2}. \tag{67}$$

Substitution of (61) and (64) in this yields

$$\eta = -\frac{1}{T - T_e}\rho \frac{d^*\epsilon}{dt} - \frac{h^i \partial_i T}{(T - T_e)2}. \tag{68}$$

From (62) and (63), we have the theorem.

We should note that our result coincides with the so called *Clausius–Duhem inequality* of continuum thermodynamics [33, 34] when T_e is put to zero. If, in addition, the *phenomenological equations*

$$\frac{d^*\epsilon}{dt} = -\lambda(T - T_e), \quad h^i = -\kappa \partial_i T, \tag{69}$$

where λ and κ are positive constants, are assumed, then (66) is automatically satisfied and we have a complete set of equations to solve for non-equilibrium flows. However, it seems difficult to provide physical justification for (69), and the problem is left to future investigation.

4. CONCLUDING REMARKS

We have presented a micropolar continuum model for the flow of granular materials. Equations of motion are obtained by the continuum-mechanical conservation laws, and constitutive equations are deduced by statistical consideration of particle models. We have discussed the equivalence of the continuum description and the particulate description, and obtained equivalent dissipative stresses which satisfy the minimum dissipation principle. Two flow regimes have been considered. For the slow flow regime, we have shown the perfectly plastic nature of the material. For the fast flow regime, we have adopted notions of thermodynamics and presented a consistent description of flows in "local equilibrium". It has been shown that the material exhibits the *normal stress effects*. Our results are similar to those of Bagnold but have a form of three-dimensional tensor equations. The "thermal dilatation" of the flow is analyzed to see the normal stress effects. Finally, we have obtained an "entropy" formulation which is an extension of classical continuum thermodynamics.

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