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A MICROPOLAR CONTINUUM MODEL FOR VIBRATING GRID FRAMEWORKS

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Abstract—A principle of converting a discrete system of grid frameworks to an equivalent micropolar continuum model is given with the degree of approximation taken into consideration. A micropolar continuum is then defined in the form of higher order extension. In order to supplement defects of previous theories, a complex-valued micropolar continuum model is constructed for grid frameworks vibrating with an arbitrary frequency by means of the variational principle related to the average energy of the system. An analysis of wave propagation reveals the existence of high frequency waves. The accuracy of solutions is also investigated.

1. INTRODUCTION

IN FRAME analysis, the displacements and the rotations of joints are taken as unknown variables, and the equations of equilibrium are solved to determine them. Usually, electronic computers are used, but the calculation becomes expensive, if the structure is large. Hence, a suitable technique for approximation is desired. For this purpose, continuous approximation has been frequently used. In the beginning, finite difference equations governing the structure were directly replaced by corresponding differential equations by means of Taylor expansion (Renton[1, 2]). Then, the fact that such continuous approximation means nothing but defining new continua was recognized, and study of the underlying continua as a new physical entity began in connection with recent development of the so-called mechanics of generalized or structured continua[3]. Kanatani[4], for example, studied mechanics of continua with projective microstructure as a continuous model for a truss. On the other hand, Banks and Sokolowski[5] and Askar and Cakmak[6] found that the micropolar continuum of Eringen[7] was suitable as a continuous model for grid frameworks. Later, some inadequacy in their works was pointed out by Bažant and Christensen[8], who gave a consistent formulation of the problem on the basis of continuum mechanics.

In order to convert a discrete system to an equivalent continuum model, Taylor expansions of the relative displacements and rotations of joints, with only a few beginning terms retained, were used in [5, 6, 8]. This procedure, however, does not lead to adequate equations, unless the degree of approximation is properly taken into consideration. In this paper, we shall first give a suitable principle of continuous approximation and define the order of approximation. The equations of equilibrium are then derived by means of the variational principle of virtual work. The micropolar continuum thus defined is not in the usual form but in the form of higher order extension.

Continuous approximation is effective especially in dynamic problems, because the exact calculation is more difficult in dynamic cases than in static ones. It is shown by Sun and Yang[9] that the micropolar continuum model can be employed in dynamic problems with great ease. However, their model cannot describe such an important phenomenon as resonance, because they made an implicit assumption that static characteristics of member beams are valid even when they are in motion. It can be shown that general equations of motion of an equivalent continuum cannot be obtained without this assumption. In this paper, we shall circumvent this difficulty by restricting our attention to vibration alone. We shall give a complex-valued micropolar continuum model for vibrating grid frameworks applicable to vibrations of any frequencies. The equations of vibration are derived by means of the variational principle related to the average energy of the system. Wave propagation is analyzed, and the existence of high frequency waves, which were missing in the analysis in [9], is shown. Sun and Yang[9] showed that their analysis was in good agreement with several sample solutions calculated by the finite element method in the range of low frequency. We shall show that our result is in fairly good agreement with the exact solution in a wide range of frequency, if the order of continuous approximation is properly taken.

2. THE ORDER OF CONTINUOUS APPROXIMATION AND HIGHER ORDER MICROPOLAR CONTINUA

In continuum mechanics, the form of potential energy expressed in terms of field variables is of fundamental importance. Given the potential energy form, we can derive the equations of equilibrium by means of the principle of virtual work. In a grid framework, potential energy is stored in each of the member beams by elongation, bending and twisting. Deformation of a beam is completely specified by the displacements and rotations of its both ends. Let $u_i(I, J, K)$ and $\phi_{ji}(I, J, K)$ be the displacement and the rotation of the joint labeled as (I, J, K), respectively. The rotation ϕ_{ji} is an skewsymmetric tensor, and we, henceforth, adopt the rule of summation convention. The coordinate system is always Cartesian, so that we do not make any distinction between covariant and contravariant components of vectors and tensors. The total potential energy of the system is expressed in the form

$$U = \sum_{I,J,K} \epsilon(\Delta_j u_i, \phi_{ji}, \Delta_k \phi_{ji}), \tag{1}$$

where Δ_j denotes the finite difference operator in the j-direction (see Fig. 1). An equivalent continuum model is obtained by replacing the finite difference terms by their Taylor series expansions

$$\begin{aligned} \Delta_j u_i &= h_j \partial_j u_i + \frac{1}{2} h_j^2 \partial_j^2 u_i + \dots, \\ \Delta_k \phi_{ji} &= h_k \partial_k \phi_{ji} + \frac{1}{2} h_k^2 \partial_k^2 \phi_{ji} + \dots, \end{aligned} \tag{2}$$

where h_i is the length of beams lying in the i-direction, and ∂_i denotes $\partial/\partial x^i$. Only the first terms of (2) are retained in [5, 6]. This is, however, not an adequate approximation, because the form of ϵ in (1) is quadratic in its arguments. The validity of omitting succeeding terms comes from the fact that they are quantities of higher order smallness. Then the square $(h_k \partial_k \phi_{ji})^2$, for example, is of the second order smallness, and it is comparable with the omitted term $(1/2) h_k^2 \partial_k^2 \phi_{ji}$ multiplied by the zeroth order term ϕ_{ji}. To avoid this inconsistency, we adopt the following procedure; substitute formally the infinite series (2) into (1) and omit those terms whose orders are higher than N. Let us call this approximation the Nth *order continuous approximation*, and that used in [5, 6] the *simple approximation*. Both the second order approximation and the simple approximation lead to second order differential equations, but several terms in those of the former are missing in those of the latter. Bažant and Christensen [8] noticed this fact and gave a complete set of equations, taking account of those terms. A consistent principle is now given by the above stated method of approximation.

Now, we can derive field equations by the principle of virtual work. After the above mentioned procedure, the total potential energy of the system is approximated by the continuous form.

$$U = \int \epsilon(\partial_j u_i, \partial_k \partial_j u_i, \dots, \phi_{ji}, \partial_k \phi_{ji}, \dots) \, \mathrm{d}V. \tag{3}$$

In the second approximation the increment δU of the energy for virtual displacement δu_i and virtual rotation $\delta \phi_{ji}$ becomes

$$\delta U = \int [\sigma^{ji} \partial_j \delta u_i + \theta^{kji} \partial_k \partial_j \delta u_i + \frac{1}{2} \tau^{ji} \partial \phi_{ji} + \frac{1}{2} \mu^{kji} \partial_k \delta \phi_{ji} + \frac{1}{2} \pi^{lkji} \partial_l \partial_k \delta \phi_{ji}] \, \mathrm{d}V, \tag{4}$$

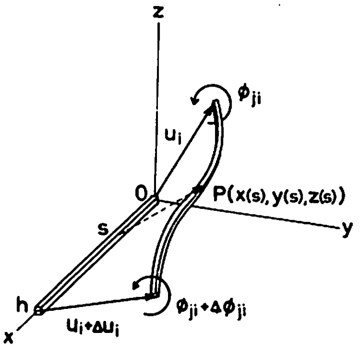

Fig. 1. Deformed and undeformed configurations of a member beam.

where $\sigma^{ji} = \partial\epsilon/\partial(\partial_j u_i)$, etc. The principle of objectivity [10] demands that the increment of potential energy should vanish for virtual rigid rotation $\delta\phi_{ji} = \partial_{[j}\delta u_{i]}$. (The brackets [ ] denote the skewsymmetric part of components. The symmetric par is denoted by ( ).) Hence, we conclude that

$$\frac{1}{2}\pi^{ji} = -\sigma^{[ji]}. \tag{5}$$

The virtual work done by external forces is

$$\delta W = \int b^i \delta u_i \, \mathrm{d}V + \oint \left[t^i \delta u_i + \frac{1}{2} m^{ji} \delta\phi_{ji} \right] \mathrm{d}S, \tag{6}$$

where b^i is the force acting on joints per unit volume, and t^i and m^{ji} are, respectively, the surface traction and the surface moment per unit area. Then the principle of virtual work $\delta U = \delta W$ yields, upon integration of (4) by parts, equations of equilibrium

$$\begin{aligned} &\partial_j\sigma^{ji} - \partial_k\partial_j\theta^{kji} + b^i = 0, \\ &\partial_k\mu^{kji} - \partial_l\partial_k\pi^{lkji} + 2\sigma^{[ji]} = 0, \end{aligned} \tag{7}$$

with boundary conditions

$$\begin{aligned} t^i &= n_j\sigma^{ji} - 2n_{(k}\partial_{j)}\theta^{kji} + n_k n_j \partial_n \theta^{kji} - \left(2n_k n_j \sum_\alpha 1/R_\alpha - \sum_\alpha l_k^{(\alpha)} l_j^{(\alpha)}/R_\alpha\right)\theta^{kji}, \\ m^{kji} &= n_k\mu^{kji} - 2n_{(l}\partial_{k)}\pi^{lkji} + n_l n_k \partial_n \theta^{lkji} - \left(2n_l n_k \sum_\alpha 1/R_\alpha - \sum_\alpha l_l^{(\alpha)} l_k^{(\alpha)}/R_\alpha\right)\pi^{lkji}, \\ & n_k n_j \theta^{kji} = 0, \quad n_l n_k \pi^{lkji} = 0, \end{aligned} \tag{8}$$

where n_i is the unit normal to the boundary surface, $l_i^{(\alpha)} (\alpha = 1, 2)$ is the principal direction of the surface, and $1/R_\alpha$ is the corresponding principal curvature. Differentiation along n_i is denoted by ∂_n. The constitutive equations are given by

$$\begin{aligned} &\sigma^{11} = \kappa_i h_1 \partial_1 u_1, \qquad \sigma^{12} = \frac{\nu_{12}}{h_1}(\partial_1 u_2 - \phi_{12}) - \frac{1}{2} h_1 \partial_1 \phi_{12}, \\ &\mu^{112} = \frac{\nu_{12}}{2}\left(\phi_{12} - \partial_1 u_2 + \frac{3}{2} h_1 \partial_1 \phi_{12}\right), \qquad \mu^{123} = \frac{\nu_{11}}{2} h_1 \partial_1 \phi_{23}, \\ &\theta^{112} = -\frac{\nu_{12}}{2}\phi_{12}, \qquad \pi^{1112} = \frac{\nu_{12}}{4} h_1^2 \phi_{12}, \text{etc.,} \end{aligned} \tag{9}$$

$$\kappa_1 \equiv E_1 A_1/V, \quad \nu_{12} \equiv 12 E_1 I_{12}/V, \quad \nu_{11} \equiv 2G_1 J_1/V, \quad V \equiv h_1 h_2 h_3,$$

where $E_i A_i$ and $G_i J_i$ (not summed) are respectively the longitudinal and the torsional stiffnesses of a beam lying in the i-direction, while $E_j I_{ji}$ (not summed) is the bending stiffness of a beam lying in the j-direction bent in the j–i plane. The other components are obtained by permutation of indices. Terms like θ^{kji} and π^{lkji} do not appear in the usual theory of micropolar continua [7], and these correspond to the terms of which Bažant and Christensen [8] took special account. Equations (7) are considered to define a micropolar continuum in the form of higher order extension.

In the case of higher order approximation, we can similarly obtain the field equations as follows. The increment δU of potential energy becomes, upon integration by parts,

$$\delta U = \int [(\delta U/\delta u_i)\delta u_i + (\delta U/\delta\phi_{ji})\delta\phi_{ji}] \, \mathrm{d}V + \oint [\text{surface terms}] \, \mathrm{d}S \tag{10}$$

where $\delta U/\delta u_i$ and $\delta U/\delta \phi_{ji}$ are the functional derivatives defined by

$$\delta U/\delta u_i = -\partial_j(\partial\epsilon/\partial(\partial_j u_i)) + \partial_k\partial_j(\partial\epsilon/\partial(\partial_k\partial_j u_i)) - \cdots, \qquad \delta U/\delta\phi_{ji} = \partial\epsilon/\partial\phi_{ji} - \partial_k(\partial\epsilon/\partial(\partial_k\phi_{ji})) + \cdots, \tag{11}$$

respectively. Then, the equations of equilibrium are

$$\delta U/\delta u_i = b^i, \quad \delta U/\delta \phi_{ji} = 0. \tag{12}$$

The boundary conditions are given by the surface integral terms.

3. EQUATIONS OF MOTION OF GRID FRAMEWORKS

Consider a moving beam as shown in Fig. 1. The kinetic energy of the beam is

$$\frac{1}{2}\int_0^h \rho A[\dot{x}(s)^2 + \dot{y}(s)^2 + \dot{z}(s)^2]\,\mathrm{d}s, \tag{13}$$

where ρA is the line density and $\dot{}$ denotes $\mathrm{d}/\mathrm{d}t$. In an equivalent continuum, the kinetic energy of a beam must be expressed only in terms of the variables at the both ends. Sun and Yang[9] implicitly assumed static bending and expressed (13) in terms of v_i, Δv_i, ω_{ji} and $\Delta\omega_{ji}$, where $v_i = \dot{u}_i$ and $\omega_{ji} = \dot{\phi}_{ji}$. Using their result and following our procedure of continuous approximation, we can express the total kinetic energy of the system in the form

$$K = \frac{1}{2}\int k(v_i, \partial_j v_i, \ldots \omega_{ji}, \partial_k \omega_{ji}, \ldots)\,\mathrm{d}V. \tag{14}$$

Differentiating this by the time, and integrating it by parts, we obtain the following form.

$$\begin{aligned}\frac{\mathrm{d}K}{\mathrm{d}t} = &\int [\overline{p^i(v_j, \partial_k v_j, \ldots, \omega_{kj}, \partial_l \omega_{kj}, \ldots)v_i} + \overline{\frac{1}{2} l^{ji}(v_k, \partial_l v_k, \ldots, \omega_{lk}, \partial_m \omega_{lk}, \ldots)\omega_{ji}}]\,\mathrm{d}V \\ &+ \oint [\text{surface terms}]\,\mathrm{d}S.\end{aligned} \tag{15}$$

The linear forms p^i and l^{ji} are the equivalent momentum density and the equivalent angular momentum density respectively. According to D'Alembert's principle, the equations of motion are obtained by adding the inertia force $-\dot{p}^i$ and the inertia torque $-\dot{l}^{ji}$ to the equations of equilibrium. Hamilton's principle is used in [9], but it is equivalent to our procedure. We call the continuum model thus obtained the *quasistatic* model, for it is valid only for slow motion. In the first order approximation, we obtain the equations of motion as follows

$$\begin{aligned}\partial_j \sigma^{ji} + b^i &= \rho \dot{v}_i - \gamma^j \partial_j \dot{\omega}_{ji}, \\ \partial_k \mu^{kji} + 2\sigma^{[ji]} &= I^{ji}\dot{\omega}_{ji} + J^j \partial_j \dot{v}_i - J^i \partial_i \dot{v}_j,\end{aligned} \tag{16}$$

where

$$\begin{aligned}&\sigma^{11} = 0, \qquad \sigma^{12} = -\frac{\nu_{12}}{h_1}\phi_{12}, \text{ etc.,} \\ &\mu^{123} = 0, \qquad \mu^{112} = \frac{\nu_{12}}{2}\phi_{12}, \text{ etc.,} \\ &\rho = \sum_i M_i/V, \qquad \gamma^i = \frac{3}{105} M_i h_i^2/V, \\ &I^{ji} = (M_j h_j^2 + M_i h_i^2)/V, \qquad J^i = \frac{13}{210} M_i h_i^2/V,\end{aligned} \tag{17}$$

and M_i is the mass of beams lying in the i-direction.

4. THE VIBRATION FIELD OF GRID FRAMEWORKS

In the previous section it was shown that, without assuming static bending the kinetic energy couldn't be expressed in terms of the variables at joints. Hence, such an important phenomenon as resonance cannot be described by the quasistatic continuum model. We shall now circumvent this difficulty by restricting our attention to vibration alone. Then we can make use of the variational principle related to the average energy of the system to obtain a complex-valued micropolar continuum model for vibrating grid frameworks.

Let x^α ($\alpha = 1, 2, 3, \ldots, n$) be the generalized coordinate of a linear mechanical system. The equation of motion has the form

$$m_{\beta\alpha}\ddot{x}^\beta + k_{\beta\alpha}x^\beta + d_{\beta\alpha}\dot{x}^\beta = Q_\alpha, \tag{18}$$

where $m_{\beta\alpha}$, $k_{\beta\alpha}$ and $d_{\beta\alpha}$ are the mass matrix, the stiffness matrix and the damping matrix of the system respectively, while Q_α is the force acting against the generalized coordinate x^α. In vibration analysis, the harmonic mode x^α, $Q_\alpha \propto e^{i\omega t}$ ($i = \sqrt{-1}$) is investigated. The equation of vibration of the system is

$$-\omega^2 m_{\beta\alpha}x^\beta + k_{\beta\alpha}x^\beta + i\omega d_{\beta\alpha}x^\beta = Q_\alpha. \tag{19}$$

Now x^α and Q^α are complex quantities. We now give a kind of variational principle which leads directly to the complex expression (19). Consider the time average of the kinetic energy K, the potential energy U and the dissipation function F. We obtain the Hermitian forms

$$\langle K\rangle = \frac{\omega^2}{4} m_{\beta\alpha}x^\alpha\overline{x^\beta}, \qquad \langle U\rangle = \frac{1}{4} k_{\beta\alpha}x^\alpha\overline{x^\beta}, \qquad \langle F\rangle = \frac{\omega^2}{4} d_{\beta\alpha}x^\alpha\overline{x^\beta}, \tag{20}$$

where $\overline{x^\alpha}$ is the complete conjugate of x^α. Then the equation of vibration is given by

$$\frac{\partial S}{\partial \overline{x^\alpha}} = Q_\alpha, \qquad S \equiv \frac{1}{4}(\langle U\rangle - \langle K\rangle + (i/\omega)\langle F\rangle). \tag{21}$$

The number n of independent variables is irrelevant in this formulation, so that this relation must also hold in the limit of continuum, in which partial derivatives must be replaced by functional derivatives with respect to field variables.

Consider a vibrating beam shown in Fig. 2, where u, Δu, v, Δv, ϕ and $\Delta\phi$ may be complex. If the longitudinal vibration of the beam is neglected, we can put

$$x(s) = \left(u + \frac{s}{l}\Delta u\right)e^{i\omega t}, \tag{22}$$

and $y(s)$ is determined by

$$EIy''''(s) + \eta\dot{y}(s) + \rho A\ddot{y}(s) = 0, \tag{23}$$

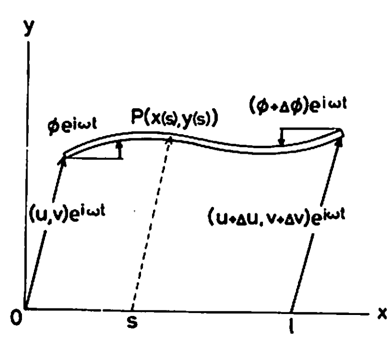

Fig. 2. Vibration of a member beam.

where η is the damping coefficient. Putting $y \propto e^{i\omega t}$, we have

$$y''''(s) = \lambda^4 y(s), \qquad \lambda \equiv \sqrt[4]{[(\rho A\omega^2 - i\omega\eta)/EI]}. \tag{24}$$

The boundary conditions are

$$\begin{aligned} y(0) &= v\, e^{i\omega t}, & y'(0) &= \phi\, e^{i\omega t}, \\ y(l) &= (v + \Delta v)\, e^{i\omega t}, & y'(l) &= (\phi + \Delta\phi)\, e^{i\omega t}. \end{aligned} \tag{25}$$

The solution is determined in the following linear form in v, Δv, ϕ and $\Delta\phi$.

$$y(s) = (v\psi_1(s) + \Delta v\psi_2(s) + l\phi\psi_3(s) + l\Delta\phi\psi_4(s))\, e^{i\omega t}. \tag{26}$$

Here ψ_α's are complex-valued non-dimensional functions of s (see Appendix). If the strain energy due to elongation and twisting is neglected, the average strain energy of bending is

$$\langle U\rangle = \frac{1}{4}\int_0^l EIy''(s)\overline{y''(s)}\, ds = \frac{EI}{4l}(v/l, \Delta v/l, \phi, \Delta\phi)[C_{\beta\alpha}(\lambda l)]\begin{bmatrix} \bar{v}/l \\ \Delta\bar{v}/l \\ \bar{\phi} \\ \Delta\bar{\phi} \end{bmatrix}, \tag{27}$$

where $C_{\beta\alpha}$ is the following non-dimensional real symmetric matrix.

$$C_{\beta\alpha}(\lambda l) = \frac{|\lambda l|^4}{l}\int_0^l \psi''_\beta(s)\overline{\psi''_\alpha(s)}\, ds \qquad (\alpha, \beta = 1, 2, 3, 4). \tag{28}$$

The average kinetic energy $\langle K\rangle$ and the average dissipation function $\langle F\rangle$ are determined in the same manner. They are also Hermitian forms in $u, \Delta u, v, \Delta v, \phi$ and $\Delta\phi$.

$$\begin{aligned} \langle K\rangle &= \frac{\omega^2}{4}\int_0^l \rho A(x(s)\overline{x(s)} + y(s)\overline{y(s)})\, ds = \frac{Ml^2\omega^2}{4}(u/l, \Delta u/l)\begin{bmatrix} 1 & 1/2 \\ 1/2 & 1 \end{bmatrix}\begin{bmatrix} \bar{u}/l \\ \Delta\bar{u}/l \end{bmatrix} \\ &+ \frac{Ml^2\omega^2}{4}(v/l, \Delta v/l, \phi, \Delta\phi)[M_{\beta\alpha}(\lambda l)]\begin{bmatrix} \bar{v}/l \\ \Delta\bar{v}/l \\ \bar{\phi} \\ \Delta\bar{\phi} \end{bmatrix}. \end{aligned} \tag{29}$$

$$\langle F\rangle = \frac{\omega^2}{4}\int_0^l \eta y(s)\overline{y(s)}\, ds = \frac{\eta l^3\omega^2}{4}(v/l, \Delta v/l, \phi, \Delta\phi)[M_{\beta\alpha}(\lambda l)]\begin{bmatrix} \bar{v}/l \\ \Delta\bar{v}/l \\ \bar{\phi} \\ \Delta\bar{\phi} \end{bmatrix}. \tag{30}$$

$$M_{\beta\alpha}(\lambda l) = \frac{1}{l}\int_0^l \psi_\beta(s)\overline{\psi_\alpha(s)}\, ds \qquad (\alpha, \beta = 1, 2, 3, 4). \tag{31}$$

Consider a 3-dimensional vibrating grid framework. Summing the above expressions for beams in all directions, we obtain the average kinetic energy density $\langle k\rangle$, the average potential energy density $\langle\epsilon\rangle$ and the average dissipation function density $\langle f\rangle$ in Hermitian forms. Put

$$s = \frac{1}{4}(\langle\epsilon\rangle - \langle k\rangle + (i/\omega)\langle f\rangle). \tag{32}$$

After the continuous approximation in accordance with our principle, this quantity becomes a function of the field variables and their complex conjugates. Put

$$S = \int s(u_j, \bar{u}_j, \partial_k u_j, \partial_k \bar{u}_j, \ldots, \phi_{kj}, \bar{\phi}_{kj}, \ldots)\, dV. \tag{33}$$

Taking functional derivatives with respect to the complex conjugate field variables, we obtain the following equations of vibration.

$$(\delta S/\delta \overline{u_j} =)\partial s/\partial \overline{u_j} - \partial_k(\partial s/\partial(\partial_k \overline{u_j})) + \ldots = b^j,$$
$$(\delta S/\delta \overline{\phi_{kj}} =)\partial s/\partial \overline{\phi_{kj}} - \partial_l(\partial s/\partial(\partial_l \overline{\phi_{kj}})) + \ldots = 0. \tag{34}$$

Here, we have put the force per unit volume acting on the joints to be $b^j e^{i\omega t}$. The boundary conditions are given by the surface integral terms.

We can regard the above process as a definition of a complex-valued micropolar continuum. Put

$$U = \frac{1}{4}\int \langle \epsilon \rangle \, dV, \quad K = \frac{1}{4}\int \langle k \rangle \, dV, \quad F = \frac{i}{4\omega}\int \langle f \rangle \, dV, \tag{35}$$

and consider the following variations.

$$u_j \to u_j, \quad \overline{u_j} \to \overline{u_j} + \delta\overline{u_j}, \quad \phi_{ji} \to \phi_{ji}, \quad \overline{\phi_{ji}} \to \overline{\phi_{ji}} + \delta\overline{\phi_{ji}}. \tag{36}$$

The corresponding increment of U, K and F are

$$\delta U = \int [(\delta U/\delta\overline{u_j})\delta\overline{u_j} + (\delta U/\delta\overline{\phi_{ji}})\delta\overline{\phi_{ji}}] \, dV + \int [\text{surface terms}] \, dS,$$

$$\delta K = \int [(\delta K/\delta\overline{u_j})\delta\overline{u_j} + (\delta K/\delta\overline{\phi_{ji}})\delta\overline{\phi_{ji}}] \, dV + \int [\text{surface terms}] \, dS,$$

$$\delta F = \int [(\delta F/\delta\overline{u_j})\delta\overline{u_j} + (\delta F/\delta\overline{\phi_{ji}})\delta\overline{\phi_{ji}}] \, dV + \int [\text{surface terms}] \, dS. \tag{37}$$

The functional derivatives $\delta U/\delta u_j$, etc. are all complex linear forms in $u_j, \partial_k u_j, \ldots, \phi_{kj}, \partial_l\phi_{kj}, \ldots$. We can interpret these forms as follows: $\delta U/\delta \overline{u_j}$, the complex internal force density; $\delta U/\delta u_j$, the complex internal torque density; $\delta K/\delta \overline{u_j}$, the complex momentum density; $\delta K/\delta\overline{\phi_{ji}}$, the complex angular momentum density; $\delta F/\delta u_j$, the complex dissipative force density; $\delta F/\delta\phi_{ji}$, the complex dissipative torque density.

5. WAVE PROPAGATION AND ACCURACY OF SOLUTION

Consider shearing vibration of a 2-dimensional grid framework of infinite extent. The grid is assumed to be composed of identical members of length l, stiffness EI and line density ρA. We adopt the natural units; we measure all length in terms of l, all masses in terms of ρAl, and the time in terms of $l^2\surd(\rho A/EI)$. Let us seek a 1-dimensional solution in the form $u_j = (0, v(x))\, e^{i\omega t}$, $\phi = \phi(x)\, e^{i\omega t}$ for $b^j = (0, b(x))\, e^{i\omega t}$. In the second order approximation, eqns. (34) are reduced to

$$A_1 v + A_1 v'' + C\phi' = b, \qquad -Cv' + B_1\phi + B_2\phi'' = 0, \tag{38}$$

where

$$A_1 = -\omega^2(M_{11}+1) + C_{11} + i\omega\eta M_{11},$$
$$A_2 = -\omega^2(M_{12}-M_{22}) + (C_{12}-C_{22}) + i\omega\eta(M_{12}-M_{22}),$$
$$B_1 = -2\omega^2 M_{33} + 2C_{33} + 2i\omega\eta M_{33}, \tag{39}$$
$$B_2 = -\omega^2(M_{34}-M_{44}) + (C_{34}-C_{44}) + i\omega\eta(M_{34}-M_{44}),$$
$$C = -\omega^2(M_{14}-M_{23}) + (C_{14}-C_{23}) + i\omega\eta(M_{14}-M_{23}).$$

Consider M_{11}, for example. It plays the role of the equivalent mass density of the vibrating grid framework and depends on ω and η as shown in Fig. 3. The angular frequency $\omega_0 = 22.37\surd(EI/\rho A/l^2)$ is the first characteristic angular frequency of member beams. Similarly M_{33}

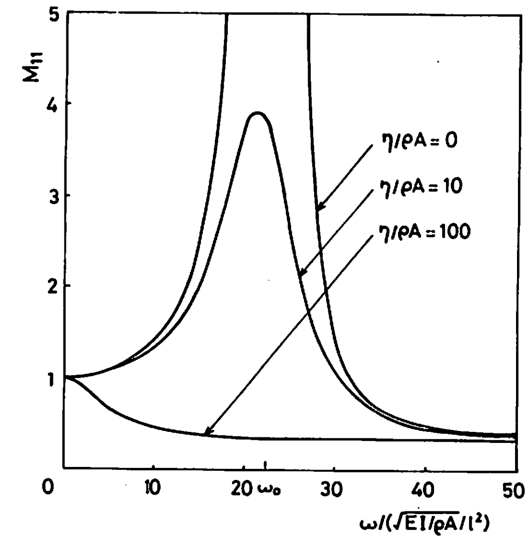

Fig. 3. Virtual mass M_{11} of a vibrating member beam.

plays the role of the equivalent moment of inertia density (Fig. 4). If all $C_{\beta\alpha}$'s and $M_{\beta\alpha}$'s in (39) are replaced by their values at $\omega = 0$ then eqns (38) are reduced to those of quasistatic *model (see* Appendix).

Since continuum models are obtained by the neglect of higher order derivatives, solutions are fairly accurate for low order modes of vibration. A systematic investigation of accuracy is possible, if we study the problem of wave propagation, for the wavelength then plays the role of the mode of vibration. Let $v(x)$ and $\phi(x)$ in (38) be proportional to e^{-ikx}. Then, we obtain a set of linear algebraic equations. Wave propagation is possible only when the determinant of the equations vanishes. Figure 5 shows the angular frequency of the wave of wave number k, including also those obtained by the first, the third and the fourth order approximations. The exact solution † is indicated by the thick solid curves in Fig. 5. We can conclude from this result that the second order approximation is accurate enough for $kl < 1$. Due to the regularity of the grid, $kl = \pi$ corresponds to the shortest possible wavelength. The situation is similar to the Brillouin zone in crystal physics. We should note here that the high frequency waves shown in Fig. 5 cannot be obtained by the quasistatic continuum model, although they are very important, especially when the structure is given a strong shock containing high frequency Fourier components.

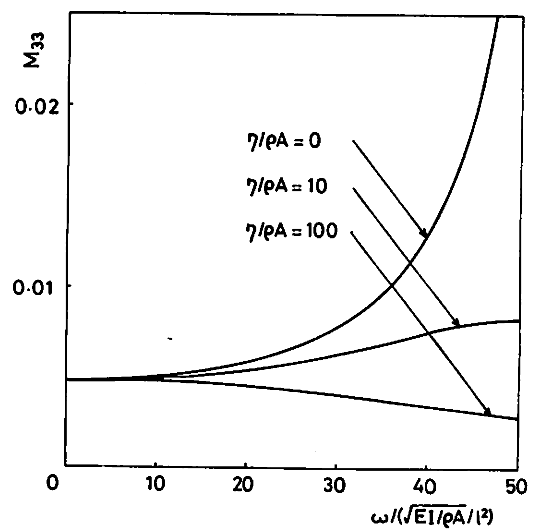

Fig. 4. Virtual moment of inertia M_{33} of a vibrating member beam.

†Sun and Yang[9] calculated the wave frequency by the quasistatic model of the simple approximation in the range of low frequency and compared the result with several numerical examples calculated by the finite element method. However, the exact solution can be obtained also by the variational principle of average energy.

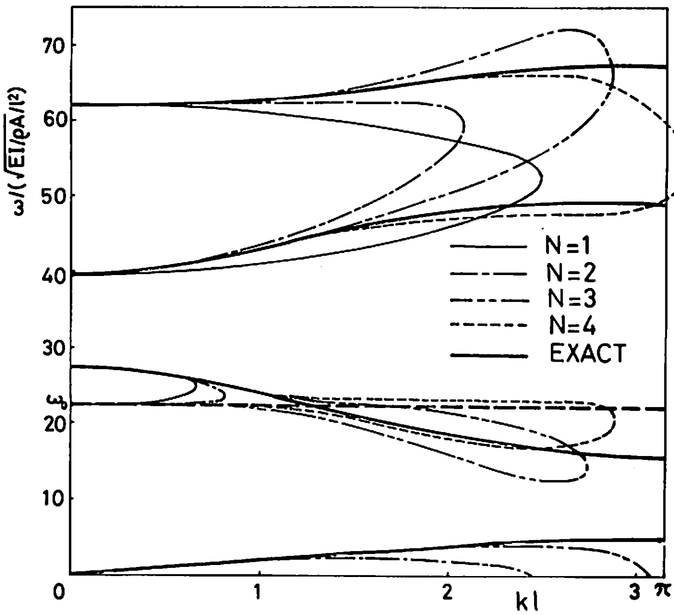

Fig. 5. Angular frequencies of traveling plane waves in the Nth order continuous approximation compared with exact solutions ($\eta = 0$).

6. CONCLUDING REMARK

We have presented a principle to convert a system of grid frameworks to an equivalent continuum model, taking the degree of approximation into consideration. As a result, a micropolar continuum is defined in the form of higher order extension. To supplement the defect of the quasistatic continuum model, we have established a complex-valued micropolar continuum model for vibrating grid frameworks. Equations of vibration are derived by means of the variational principle related to the average energy of the system. Wave propagation is analyzed, and the existence of high frequency waves is shown. The accuracy of the wave solutions is also investigated.

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APPENDIX

$$\psi_1(s) = a_1(\lambda l)(\cos\lambda s - \cosh\lambda s) + a_2(\lambda l)(\sin\lambda s - \sinh\lambda s) + \cosh\lambda s$$

$$\psi_2(s) = a_3(\lambda l)(\cos\lambda s - \cos\lambda s) + a_4(\lambda l)(\sin\lambda s - \sinh\lambda s)$$

$$\psi_3(s) = a_5(\lambda l)(\cos\lambda s - \cosh\lambda s) + a_6(\lambda l)(\sin\lambda s - \sinh\lambda s) + \frac{1}{\lambda l}\sin\lambda s$$

$$\psi_4(s) = a_7(\lambda l)(\cos\lambda s - \cosh\lambda s) + a_8(\lambda l)(\sin\lambda s - \sinh\lambda s)$$

$$a_1(\lambda l) = (\cos\lambda l - \cosh\lambda l + \sin\lambda l \sinh\lambda l - \cos\lambda l \cosh\lambda l + 1)/D(\lambda l)$$

$$a_2(\lambda l) = (\sin\lambda l + \sinh\lambda l - \cos\lambda l \sinh\lambda l - \sin\lambda l \cosh\lambda l)/D(\lambda l)$$

$$a_3(\lambda l) = (\cos \lambda l - \cosh \lambda l)/D(\lambda l)$$
$$a_4(\lambda l) = (\sin \lambda l + \sinh \lambda l)/D(\lambda l)$$
$$a_5(\lambda l) = (-\sin \lambda l + \sinh \lambda l + \sinh \lambda l \cosh \lambda l - \cos \lambda l \sin \lambda l)/\lambda l D(\lambda l)$$
$$a_6(\lambda l) = (\cos \lambda l - \cosh \lambda l - \cos \lambda l \cosh \lambda l - \sin \lambda l \sinh \lambda l + 1)/\lambda l D(\lambda l)$$
$$a_7(\lambda l) = (-\sin \lambda l + \sinh \lambda l)/\lambda l D(\lambda l)$$
$$a_8(\lambda l) = (\cos \lambda l - \cosh \lambda l)/\lambda l D(\lambda l)$$
$$D(\lambda l) = 2(1 - \cos \lambda l \cosh \lambda l)$$

$$\lim_{\lambda \to 0} M_{\beta\alpha}(\lambda l) = \begin{bmatrix} 1 & 1/2 & 0 & -1/12 \\ 1/12 & 13/35 & -3/140 & -11/210 \\ 0 & -3/140 & 1/210 & 1/420 \\ -1/12 & -11/210 & 1/420 & 1/105 \end{bmatrix}$$

$$\lim_{\lambda \to 0} C_{\beta\alpha}(\lambda l) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 12 & -12 & -6 \\ 0 & -12 & 12 & 6 \\ 0 & -6 & 6 & 4 \end{bmatrix}$$