algorithm always converges to consistent unambiguous labelings [1], [5].

## IV. Conclusion

In this correspondence, we analyzed the automata algorithm for relaxation labeling [1] for the case of symmetric compatibility functions. It is proved that starting with any initial label probabilities, the algorithm always converges to a consistent labeling. Further, all consistent unambiguous labelings are locally asymptotically stable. The algorithm analyzed in this correspondence has been employed successfully in computer vision problems such as stereopsis and object recognition [5].

## References

[1] M. A. L. Thathachar and P. S. Sastry, "Relaxation labeling with learning automata," *IEEE Trans. Pattern Anal. Machine Intell.*, vol. 8, pp. 256–268, Mar. 1986.
[2] L. S. Davis and A. Rosenfeld, "Cooperative processes in low level vision: A survey," *Artificial Intell.*, vol. 17, pp. 245–265, Aug. 1981.
[3] R. A. Hummel and S. W. Zucker, "On the foundations of relaxation labeling processes," *IEEE Trans. Pattern Anal. Machine Intell.*, vol. 5, pp. 267–287, May 1983.
[4] S. W. Zucker, Y. G. LeClerc, and J. L. Mohammed, "Continuous relaxation and local maxima selection: Conditions for equivalence," *IEEE Trans. Pattern Anal. Machine Intell.*, vol. 3, no. 2, pp. 117–126, Feb. 1981.
[5] S. Banerjee, *Stochastic Relaxation Paradigms for Low Level Vision.* Ph.D. thesis, Dept. of Elect. Eng., Indian Inst. of Sci., Bangalore, India, 1989.
[6] W. Jianhua, *The Theory of Games.* Oxford: Clarendon Press, 1988.
[7] T. Basar and G. J. Olsder, *Dynamic Noncooperative Game Theory.* New York: Academic Press, 1982.
[8] K. S. Narendra and M. A. L. Thathachar, *Learning Automata: An Introduction.* Englewood Cliffs: Prentice Hall, 1989.
[9] P. S. Sastry, "Stochastic networks for constraint satisfaction and optimization," *Sadhana*, vol. 15, pp. 251–262, Dec. 1990.
[10] H. J. Kushner, *Approximation and Weak Convergence Methods for Random Processes.* Cambridge, MA: MIT Press, 1984.
[11] K. S. Narendra and A.Annaswamy, *Stable Adaptive Systems.* Englewood Cliffs, NJ: Prentice Hall, 1989.

# Analysis of 3-D Rotation Fitting

Kenichi Kanatani

***Abstract*—Computational techniques for fitting a 3-D rotation to 3-D data are recapitulated in a refined form as minimization over proper rotations, extending three existing methods—the method of singular value decomposition, the method of polar decomposition, and the method of quaternion representation. Then, we describe the problem of 3-D motion estimation in this new light. Finally, we define the covariance matrix of a rotation and analyze the statistical behavior of errors in 3-D rotation fitting.**

***Index Terms*—3-D rotation, singular value decomposition, polar decomposition, quaternion representation, essential matrix, covariance matrix.**

## I. Introduction

In robotics applications, we often encounter the problem of computing a 3-D rigid motion that maps a set of 3-D points to another set. This problem typically occurs when 3-D data are obtained by stereo, range sensing, tactile sensing, etc. If we compute the centroid of each set and translate them in space so that their centroids come to the coordinate origin, the remaining problem is to determine the 3-D rotation that maps the first set of orientations to the second set. Thus, all we need to do is fit a 3-D rotation to the rotated data, say by least squares.

The first analytical technique for 3-D rotation fitting was reported by Horn [2], who used the *quaternion representation.* Equivalent techniques were presented by Arun *et al.* [1], using *singular value decomposition*, and by Horn *et al.* [3], using *polar decomposition.* However, their techniques dealt with minimization over orthogonal matrixes. As a result, improper rotations (i.e., rotations of determinant $-1$) can be predicted for noisy data. Umeyama [13] made a correction to the method of Arun *et al.* [1], but his derivation, based on a variational principle and Lagrange multipliers, is very complicated and lengthy.

In this paper, we first recapitulate these techniques in a refined manner as minimization over proper rotations. Then we formulate the problem of optimal resolution of a degenerate rotation and show how this solves the problem of 3-D motion estimation from two images succinctly. Finally, we define the *covariance matrix* of rotation fitting and analyze the statistical behavior of errors.

## II. Optimal Estimation of 3-D Rotation

Consider the problem of computing a 3-D rigid motion that maps a set of 3-D points $\{(x_\alpha, y_\alpha, z_\alpha)\}$, $\alpha = 1, \cdots, N$, to another set $\{(x_\alpha{}', y_\alpha{}', z_\alpha{}')\}$, $\alpha = 1, \cdots, N$. If we compute the centroids $(\bar{x}, \bar{y}, \bar{z})$ and $(\bar{x}', \bar{y}', \bar{z}')$ of the two sets and translate them in space so that their centroids come to the coordinate origin $O$, the remaining problem is to determine the 3-D rotation that maps the first set of orientations to the second set [2].

Consider the following problem:

Manuscript received October 21, 1991; revised December 7, 1992. Recommended for acceptance by Associate Editor R. Bolle.
The author is with the Department of Computer Science Gunma University, Gunma 376, Japan.
IEEE Log Number 9212236.

*Problem 1:* Given two sets of vectors $\{\mathbf{m}_\alpha\}$ and $\{\mathbf{m}'_\alpha\}$, $\alpha = 1, \cdots, N$, compute a rotation $\mathbf{R}$ such that

$$\sum_{\alpha=1}^{N} W_\alpha \|\mathbf{m}_\alpha - \mathbf{R}\mathbf{m}'_\alpha\|^2 \rightarrow \min. \tag{1}$$

Here, $W_\alpha$ is a positive weight for the $\alpha$th datum. The weights should be determined so that reliable data are given large weights whereas unreliable data are given small weights (we will discuss this later).

We use parentheses $(\cdot,\cdot)$ for vector inner product, superscript $\top$ for vector and matrix transpose, and $\mathrm{tr}(\cdot)$ for matrix trace. The left-hand side of (1) is expanded in the form

$$\sum_{\alpha=1}^{N} W_\alpha \|\mathbf{m}_\alpha\|^2 - 2\mathrm{tr}\left(\mathbf{R}^\top \sum_{\alpha=1}^{N} W_\alpha \mathbf{m}_\alpha \mathbf{m}'^\top_\alpha\right) + \sum_{\alpha=1}^{N} W_\alpha \|\mathbf{m}'_\alpha\|^2. \tag{2}$$

Hence, if the *correlation matrix* $\mathbf{K}$ between $\{\mathbf{m}_\alpha\}$ and $\{\mathbf{m}'_\alpha\}$, $\alpha = 1, \cdots, N$, is defined by

$$\mathbf{K} = \sum_{\alpha=1}^{\mathbf{N}} \mathbf{W}_\alpha \mathbf{m}_\alpha \mathbf{m}'^\top_\alpha, \tag{3}$$

Problem 1 is restated as follows:

*Problem 2:* Given a correlation matrix $\mathbf{K}$, compute a rotation $\mathbf{R}$ such that

$$\mathrm{tr}(\mathbf{R}^\top \mathbf{K}) \rightarrow \max. \tag{4}$$

The following two lemmas are fundamental:

*Lemma 1:* If $\mathbf{S}$ is a semipositive definite symmetric matrix, $\mathrm{tr}(\mathbf{RS})$ is maximized over all rotations by $\mathbf{R} = \mathbf{I}$. The solution is unique if $\mathrm{rank}\mathbf{S} > 1$.

*Proof:* If $\mathbf{S}$ is semipositive definite symmetric, it has nonnegative eigenvalues $\sigma_1 \geq \sigma_2 \geq \sigma_3$ ($\geq 0$), and the corresponding eigenvectors $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$ can be chosen to be mutually orthogonal unit vectors. Hence, $\mathbf{S}$ is expressed in the following form (the *spectral decomposition* [9]):

$$\mathbf{S} = \sum_{i=1}^{3} \sigma_i \mathbf{u}_i \mathbf{u}_i^\top. \tag{5}$$

Then,

$$\mathrm{tr}(\mathbf{RS}) = \sum_{i=1}^{3} \sigma_i \mathrm{tr}(\mathbf{R}\mathbf{u}_i\mathbf{u}_i^\top) = \sum_{i=1}^{3} \sigma_i (\mathbf{R}\mathbf{u}_i, \mathbf{u}_i). \tag{6}$$

By the *Schwartz inequality* $(\mathbf{R}\mathbf{u}_i, \mathbf{u}_i) \leq \|\mathbf{R}\mathbf{u}_i\| \cdot \|\mathbf{u}_i\| = 1$, we see that

$$\mathrm{tr}(\mathbf{RS}) \leq \sum_{i=1}^{3} \sigma_i = \mathrm{tr}\mathbf{S}. \tag{7}$$

If $\mathbf{S}$ is nonsingular (i.e., $\mathrm{rank}\mathbf{S} = 3$), all eigenvalues are positive, so the equality holds if and only if $\mathbf{R}\mathbf{u}_i = \mathbf{u}_i$, $i = 1, 2, 3$. Since $\{\mathbf{u}_i\}$ is an orthonormal system, this is true if and only if $\mathbf{R} = \mathbf{I}$. If $\mathrm{rank}\mathbf{S} = 2$, then $\sigma_1 \geq \sigma_2 > 0$ and $\sigma_3 = 0$. Hence, the equality holds if and only if $\mathbf{R}\mathbf{u}_i = \mathbf{u}_i$, $i = 1, 2$. Since $\{\mathbf{u}_i\}$ is an orthonormal system and $\mathbf{R}$ is a rotation, $\mathbf{R}\mathbf{u}_3$ is necessarily $\mathbf{u}_3$. Thus, $\mathbf{R} = \mathbf{I}$. If $\mathrm{rank}\mathbf{S} = 1$, then $\sigma_1 > 0$ and $\sigma_2 = \sigma_3 = 0$, so the equality holds as long as $\mathbf{R}\mathbf{u}_1 = \mathbf{u}_1$. In other words, any rotation around $\mathbf{u}_1$ gives a solution. If $\mathrm{rank}\mathbf{S} = 0$, then $\mathbf{S} = \mathbf{O}$; any rotation is a solution. □

The above proof is the same as that given by Arun *et al.* [1]. The following lemma is equivalent to the result of Umeyama [13].

*Lemma 2:* If $\mathbf{S}$ is a semipositive definite symmetric matrix, $\mathrm{tr}(\mathbf{R}'\mathbf{S})$ is maximized over all improper rotations for

$$\mathbf{R}' = \mathbf{I} - 2\mathbf{u}_m\mathbf{u}_m^\top, \tag{8}$$

where $\mathbf{u}_m$ is the unit eigenvector of $\mathbf{S}$ for the smallest eigenvalue. The solution is unique if $\mathrm{rank}\mathbf{S} > 1$ and the smallest eigenvalue of $\mathbf{S}$ is a simple root.

*Proof:* As in Lemma 1, let $\sigma_1 \geq \sigma_2 \geq \sigma_3$ be the eigenvalues of $\mathbf{S}$, and $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ an orthonormal system of the corresponding eigenvectors. If $\sigma_3 = 0$, the proof of Lemma 1 shows that $\mathrm{tr}(\mathbf{R}'\mathbf{S})$ is maximized by a (proper or improper) rotation $\mathbf{R}'$ such that $\mathbf{R}'\mathbf{u}_1 = \mathbf{u}_1$ and $\mathbf{R}'\mathbf{u}_2 = \mathbf{u}_2$. Since $\mathbf{R}'$ is improper, we automatically have $\mathbf{R}'\mathbf{u}_3 = -\mathbf{u}_3$. This means that $\mathbf{R}'$ is diagonal with diagonal elements $\{1, 1, -1\}$ with respect to basis $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$. Since $\mathbf{u}_1\mathbf{u}_1^\top + \mathbf{u}_2\mathbf{u}_2^\top + \mathbf{u}_3\mathbf{u}_3^\top = \mathbf{I}$ (the unit matrix), we obtain

$$\mathbf{R}' = \mathbf{u}_1\mathbf{u}_1^\top + \mathbf{u}_2\mathbf{u}_2^\top - \mathbf{u}_3\mathbf{u}_3^\top = \mathbf{I} - 2\mathbf{u}_3\mathbf{u}_3^\top. \tag{9}$$

So, assume that $\mathbf{S}$ is nonsingular. If $\mathbf{R}'$ attains the maximum of $\mathrm{tr}(\mathbf{R}'\mathbf{S})$, superimposition on $\mathbf{R}'$ of an infinitesimally small rotation in the form of $(\mathbf{I}+\epsilon\mathbf{W}+O(\epsilon^2))\mathbf{R}'$ yields zero perturbation to a first approximation in $\epsilon$, where $\mathbf{W}$ is an arbitrary antisymmetric matrix. Since

$$\mathrm{tr}((\mathbf{I}+\epsilon\mathbf{W}+O(\epsilon^2))\mathbf{R}'\mathbf{S}) = \mathrm{tr}(\mathbf{R}'\mathbf{S}) + \epsilon\mathrm{tr}(\mathbf{W}\mathbf{R}'\mathbf{S}) + O(\epsilon^2), \tag{10}$$

the term $\mathrm{tr}(\mathbf{W}(\mathbf{R}'\mathbf{S}))$ must vanish for any antisymmetric matrix $\mathbf{W}$. This occurs if and only if $\mathbf{R}'\mathbf{S}$ is a symmetric matrix, namely $\mathbf{R}'\mathbf{S} = (\mathbf{R}'\mathbf{S})^\top = \mathbf{S}\mathbf{R}'^{-1}$. Hence,

$$\mathrm{tr}(\mathbf{R}'^2\mathbf{S}) = \mathrm{tr}(\mathbf{R}'(\mathbf{R}'\mathbf{S})) = \mathrm{tr}(\mathbf{R}'\mathbf{S}\mathbf{R}'^{-1}) = \mathrm{tr}(\mathbf{R}'^{-1}\mathbf{R}'\mathbf{S}) = \mathrm{tr}\mathbf{S}. \tag{11}$$

Since $\mathbf{S}$ is nonsingular and $\mathbf{R}^2$ is proper, Lemma 1 implies that this holds if and only if $\mathbf{R}'^2 = \mathbf{I}$, i.e., $\mathbf{R}' = \mathbf{R}'^{-1}$. Hence, $\mathbf{R}'\mathbf{S} = \mathbf{S}\mathbf{R}'^{-1} = \mathbf{S}\mathbf{R}'$, i.e., $\mathbf{R}'$ *commutes* with $\mathbf{S}$. This means that $\mathbf{R}'$ and $\mathbf{S}$ are diagonalized at the same time; the orthonormal system $\{\mathbf{u}_i\}$ of eigenvectors of $\mathbf{S}$ can be chosen to be the eigenvectors of $\mathbf{R}'$ as well. Since $\mathbf{R}'$ is orthogonal, its real eigenvalues are $\pm 1$. Hence, $\mathbf{R}'\mathbf{u}_i = \pm\mathbf{u}_i$, $i = 1, 2, 3$, but $\mathbf{R}'$ is improper, so the three eigenvalues cannot be all 1. Since $\sigma_1 \geq \sigma_2 \geq \sigma_3 > 0$, (6) implies that the maximum of $\mathrm{tr}(\mathbf{R}'\mathbf{S})$ is attained when $(\mathbf{R}'\mathbf{u}_1, \mathbf{u}_1) = (\mathbf{R}'\mathbf{u}_2, \mathbf{u}_3) = 1$ and $(\mathbf{R}'\mathbf{u}_3, \mathbf{u}_2) = -1$, i.e., $\mathbf{R}'\mathbf{u}_1 = \mathbf{u}_1$, $\mathbf{R}'\mathbf{u}_2 = \mathbf{u}_2$, and $\mathbf{R}'\mathbf{u}_3 = -\mathbf{u}_3$. Thus, $\mathbf{R}'$ is diagonal with diagonal elements $\{1, 1, -1\}$ with respect to basis $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ and hence is given by (9). The uniqueness condition is obtained in the say way as in Lemma 1. □

The first procedure for solving Problem 2 is to decompose the correlation matrix $\mathbf{K}$ into the form

$$\mathbf{K} = \mathbf{V}\mathbf{\Lambda}\mathbf{U}^\top, \quad \mathbf{\Lambda} = \begin{pmatrix} \sigma_1 & & \\ & \sigma_2 & \\ & & \sigma_3 \end{pmatrix}, \quad \sigma_1 \geq \sigma_2 \geq \sigma_3 \geq 0, \tag{12}$$

where $\mathbf{V}$ and $\mathbf{U}$ are orthogonal matrixes. This decomposition is called the *singular value decomposition*, and $\sigma_1$, $\sigma_2$, and $\sigma_3$ the *singular values*. The number of nonzero singular values is the *rank* of $\mathbf{K}$, which is equal to the number of linearly independent columns (or rows) of $\mathbf{K}$. The following theorem is mathematically equivalent to Umeyama's extension [13] of the method of Arun *et al.* [1].

*Theorem 1:* If $\mathbf{K} = \mathbf{V}\mathbf{\Lambda}\mathbf{U}^\top$ is the singular value decomposition, $\mathrm{tr}(\mathbf{R}^\top\mathbf{K})$ is maximized over all rotations by

$$\mathbf{R} = \mathbf{V}\begin{pmatrix} 1 & & \\ & 1 & \\ & & \det(\mathbf{V}\mathbf{U}^\top) \end{pmatrix}\mathbf{U}^\top. \tag{13}$$

The solution is unique if $\mathrm{rank}\mathbf{K} > 1$ and $\det(\mathbf{V}\mathbf{U}^\top) = 1$, or if $\mathrm{rank}\mathbf{K} > 1$ and the minimum singular value is a simple root.

*Proof:* If $\mathbf{K} = \mathbf{V}\mathbf{\Lambda}\mathbf{U}^\top$, we have

$$\mathrm{tr}(\mathbf{R}^\top\mathbf{K}) = \mathrm{tr}(\mathbf{R}^\top\mathbf{V}\mathbf{\Lambda}\mathbf{U}^\top) = \mathrm{tr}((\mathbf{U}^\top\mathbf{R}^\top\mathbf{V})\mathbf{\Lambda}). \quad (14)$$

Since $\mathbf{V}$ and $\mathbf{U}^\top$ are orthogonal matrixes, $\det(\mathbf{V}\mathbf{U}^\top)$ is either 1 or $-1$. If it is 1, the matrix $\mathbf{U}^\top\mathbf{R}^\top\mathbf{V}$ ranges over all proper rotations as $\mathbf{R}$ ranges over all rotations. Since $\mathbf{\Lambda}$ is a semipositive definite symmetric matrix, Lemma 1 implies that (14) attains its maximum when $\mathbf{U}^\top\mathbf{R}^\top\mathbf{V} = \mathbf{I}$ or $\mathbf{R} = \mathbf{V}\mathbf{U}^\top$. If $\det(\mathbf{V}\mathbf{U}^\top) = -1$, the matrix $\mathbf{U}^\top\mathbf{R}^\top\mathbf{V}$ ranges over all improper rotations as $\mathbf{R}$ ranges over all rotations. By Lemma 2, the maximum is attained when $\mathbf{U}^\top\mathbf{R}^\top\mathbf{V} = \mathbf{\Lambda}'$ or $\mathbf{R} = \mathbf{V}\mathbf{\Lambda}'\mathbf{U}^\top$, where $\mathbf{\Lambda}'$ is the diagonal matrix with diagonal elements $\{1, 1, -1\}$ in this order. The uniqueness condition also follows from Lemmas 1 and 2. □

The second procedure for solving Problem 2 is to decompose the correlation matrix $\mathbf{K}$ into the form

$$\mathbf{K} = \mathbf{V}\mathbf{S} = \mathbf{S}'\mathbf{V}, \quad (15)$$

where $\mathbf{V}$ is an orthogonal matrix, while $\mathbf{S}$ and $\mathbf{S}'$ are semipositive definite symmetric matrixes. This decomposition is known as the *polar decomposition*. The following theorem is an extension of the method of Horn *et al.* [3].

*Theorem 2:* If $\mathbf{K} = \mathbf{V}\mathbf{S} = \mathbf{S}'\mathbf{V}$ is the polar decomposition, $\mathrm{tr}(\mathbf{R}^\top\mathbf{K})$ is maximized over all rotations by

$$\mathbf{R} = \mathbf{V}(\mathbf{I}+(\det\mathbf{V}-1)\mathbf{u}_m\mathbf{u}_m^\top) = (\mathbf{I}+(\det\mathbf{V}-1)\mathbf{v}_m\mathbf{v}_m^\top)\mathbf{V}, \quad (16)$$

where $\mathbf{u}_m$ and $\mathbf{v}_m$ are the unit eigenvectors of $\mathbf{S}$ and $\mathbf{S}'$, respectively, for the smallest eigenvalue. The solution is unique if $\mathrm{rank}\mathbf{K} > 1$ and $\det\mathbf{V} = 1$, or if $\mathrm{rank}\mathbf{K} > 1$ and the smallest eigenvalue of $\mathbf{S}$ (and of $\mathbf{S}'$) is a simple root.

*Proof:* If $\mathbf{K} = \mathbf{V}\mathbf{S}$, we have

$$\mathrm{tr}(\mathbf{R}^\top\mathbf{K}) = \mathrm{tr}((\mathbf{R}^\top\mathbf{V})\mathbf{S}). \quad (17)$$

Since $\mathbf{V}$ is an orthogonal matrix, $\det\mathbf{V} = \pm 1$. If $\det\mathbf{V} = 1$, the matrix $\mathbf{R}^\top\mathbf{V}$ ranges over all proper rotations as $\mathbf{R}$ ranges over all rotations. By Lemma 1, (17) attains its maximum when $\mathbf{R}^\top\mathbf{V} = \mathbf{I}$ or $\mathbf{R} = \mathbf{V}$. If $\det\mathbf{V} = -1$, the matrix $\mathbf{R}^\top\mathbf{V}$ ranges over all improper rotations as $\mathbf{R}$ ranges over all rotations. By Lemma 2, the maximum is attained when $\mathbf{R}^\top\mathbf{V} = \mathbf{I} - 2\mathbf{u}_m\mathbf{u}_m^\top$ or $\mathbf{R} = \mathbf{V}(\mathbf{I} - 2\mathbf{u}_m\mathbf{u}_m^\top)$. The same argument holds for $\mathbf{K} = \mathbf{S}'\mathbf{V}$. The uniqueness condition also follows from Lemmas 1 and 2. □

The third method is based on the well-known fact that for any rotation matrix $\mathbf{R}$, there exist four numbers $q_0$, $q_1$, $q_2$, and $q_3$ such that $q_0^2 + q_1^2 + q_2^2 + q_3^2 = 1$ and (18), shown at the bottom of the page, applies. Conversely, any four numbers $q_0$, $q_1$, $q_2$, and $q_3$ such that $q_0^2 + q_1^2 + q_2^2 + q_3^2 = 1$ define a rotation by this equation. This fact is known as the *quaternion representation* of 3-D rotation [5]. The following theorem summarizes the method of Horn [2].

*Theorem 3:* Given correlation matrix $\mathbf{K}$, define a four-dimensional symmetric matrix as shown in (19) at the bottom of the page. Let $\hat{\mathbf{q}}$ be the four-dimensional unit eigenvector of $\hat{\mathbf{K}}$ for the largest eigenvalue. Then, $\mathrm{tr}(\mathbf{R}^\top\mathbf{K})$ is maximized by the rotation represented by $\hat{\mathbf{q}}$. The solution is unique if the largest eigenvalue of $\hat{\mathbf{K}}$ is a simple root.

*Proof:* Recall the relationship $\mathrm{tr}(\mathbf{R}^\top\mathbf{K}) = \sum_{\alpha=1}^N W_\alpha(\mathbf{m}_\alpha, \mathbf{R}\mathbf{m}'_\alpha)$ for $\mathbf{K} = \sum_{\alpha=1}^N W_\alpha \mathbf{m}_\alpha \mathbf{m}'^\top_\alpha$. From the quaternion representation (18), the $i$th component of vector $\mathbf{R}\mathbf{m}'_\alpha$ is written as

$$(\mathbf{R}\mathbf{m}'_\alpha)_i = m'_{\alpha(i)}q_0^2 + 2\sum_{j,k=1}^3 \epsilon_{ijk} m'_{\alpha(k)} q_j q_0 + 2q_i \sum_{j=1}^3 m'_{\alpha(j)} q_j - m'_{\alpha(i)} \sum_{j=1}^3 q_j^2, \quad (20)$$

where $m'_{\alpha(i)}$ is the $i$th component of vector $\mathbf{m}'_\alpha$, and the symbol $\epsilon_{ijk}$ denotes the *Eddington epsilon*, taking the value 1 if $(ijk)$ is an even permutation of (123), $-1$ if $(ijk)$ is an odd permutation of (123), and 0 otherwise. Then, it is easy to see that

$$\sum_{\alpha=1}^N W_\alpha(\mathbf{m}_\alpha, \mathbf{R}\mathbf{m}'_\alpha) = q_0^2 \sum_{i=1}^3 K_{ii} + 2q_0 \sum_{i=1}^3 q_i \left(\sum_{j,k=1}^3 \epsilon_{ijk} K_{kj}\right) + \sum_{i,j=1}^3 q_i q_j \left(K_{ij} + K_{ji} - \delta_{ij} \sum_{k=1}^3 K_{kk}\right). \quad (21)$$

If we define a four-dimensional symmetric matrix $\hat{\mathbf{K}}$ by (19), (20) is rewritten in the form $\mathrm{tr}(\mathbf{R}^\top\mathbf{K}) = (\hat{\mathbf{q}}, \hat{\mathbf{K}}\hat{\mathbf{q}})$, where $\hat{\mathbf{q}} = (q_0, q_1, q_2, q_3)^\top$ is a four-dimensional unit vector. Hence, $\mathrm{tr}(\mathbf{R}^\top\mathbf{K})$ is maximized by the unit eigenvector $\hat{\mathbf{q}}$ of $\hat{\mathbf{K}}$ for the largest eigenvalue. □

## III. Orthogonality Reconstruction From Projection

The *orthogonal projection* of vector $\mathbf{a}$ onto the plane with unit surface normal $\mathbf{h}$ is given by $\mathbf{P_h}\mathbf{a}$, where

$$\mathbf{P_h} = \mathbf{I} - \mathbf{h}\mathbf{h}^\top. \quad (22)$$

(See. Fig. 1(a).) Let us consider the following problem (Fig. 1(b)):

*Problem 3:* Given three coplanar vectors $\{\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3\}$, compute a right-handed orthonormal system $\{\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3\}$ such that

$$\mathbf{t}_i = \mathbf{P_h}\mathbf{r}_i, \quad (23)$$

where $\mathbf{h}$ is the unit normal to the surface on which $\{\mathbf{t}_i\}$ lie.

If we define matrixes $\mathbf{R} = (\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3)$ and $\mathbf{T} = (\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3)$, the requirement $\mathbf{t}_i = \mathbf{P_h}\mathbf{r}_i$, $i = 1, 2, 3$, is equivalent to $\mathbf{T} = \mathbf{P_h}\mathbf{R}$. Let us call a matrix $\mathbf{T}$ a *degenerate rotation* if there exists a rotation $\mathbf{R}$ and a unit vector $\mathbf{h}$ such that $\mathbf{T} = \mathbf{P_h}\mathbf{R}$. We call the unit vector $\mathbf{h}$ the *axis* of $\mathbf{T}$.

$$\mathbf{R} = \begin{pmatrix} q_0^2 + q_1^2 - q_2^2 - q_3^2 & 2(q_1q_2 - q_0q_3) & 2(q_1q_3 + q_0q_2) \\ 2(q_2q_1 + q_0q_3) & q_0^2 - q_1^2 + q_2^2 - q_3^2 & 2(q_2q_3 - q_0q_1) \\ 2(q_3q_1 - q_0q_2) & 2(q_3q_2 + q_0q_1) & q_0^2 - q_1^2 - q_2^2 + q_3^2 \end{pmatrix}. \quad (18)$$

$$\hat{\mathbf{K}} = \begin{pmatrix} K_{11} + K_{22} + K_{33} & K_{32} - K_{23} & K_{13} - K_{31} & K_{21} - K_{12} \\ K_{32} - K_{23} & K_{11} - K_{22} - K_{33} & K_{12} + K_{21} & K_{31} + K_{13} \\ K_{13} - K_{31} & K_{12} + K_{21} & -K_{11} + K_{22} - K_{33} & K_{23} + K_{32} \\ K_{21} - K_{12} & K_{31} + K_{13} & K_{23} + K_{32} & -K_{11} - K_{22} + K_{33} \end{pmatrix}. \quad (19)$$

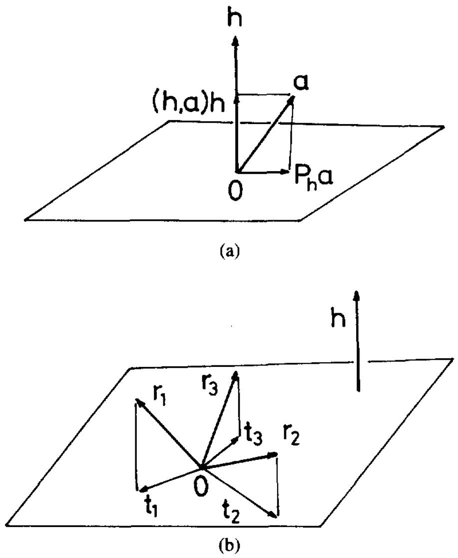



Fig. 1. (a) Orthogonal projection of a vector. (b) Orthogonal projection of an orthonormal frame.

*Theorem 4:* A matrix $\mathbf{T}$ is a degenerate rotation if and only if it has singular values 1, 1 and 0.

*Proof:* Suppose $\mathbf{T}$ is a degenerate rotation. Then, there exist an axis $\mathbf{h}$ and a rotation $\mathbf{R}$ such that $\mathbf{T} = \mathbf{P_h R}$, so

$$\mathbf{TT}^\top = \mathbf{P_h RR}^\top \mathbf{P}_\mathbf{h}^\top = \mathbf{P}_\mathbf{h}^2 = \mathbf{P_h} = \mathbf{I} - \mathbf{hh}^\top. \quad (24)$$

If unit vectors $\mathbf{u}$ and $\mathbf{v}$ are defined so that $\{\mathbf{u}, \mathbf{v}, \mathbf{h}\}$ form an orthonormal system, then $\mathbf{uu}^\top + \mathbf{vv}^\top + \mathbf{hh}^\top = \mathbf{I}$, and hence

$$\mathbf{TT}^\top = \mathbf{uu}^\top + \mathbf{vv}^\top + 0\mathbf{hh}^\top. \quad (25)$$

This means that matrix $\mathbf{T}$ has singular values $\sqrt{1}$, $\sqrt{1}$, and $\sqrt{0}$ [9].

Conversely, suppose matrix $\mathbf{T}$ has singular values 1, 1, and 0. This means that matrix $\mathbf{TT}^\top$ has eigenvalues $1^2$, $1^2$, and $0^2$ [9]. If $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{h}$ are the corresponding eigenvectors so chosen that $\{\mathbf{u}, \mathbf{v}, \mathbf{h}\}$ form an orthonormal system, we can write

$$\mathbf{TT}^\top = \mathbf{uu}^\top + \mathbf{vv}^\top + 0\mathbf{hh}^\top = \mathbf{I} - \mathbf{hh}^\top = \mathbf{P_h}. \quad (26)$$

This means that matrix $\mathbf{T}$ has the polar decomposition $\mathbf{T} = \mathbf{P_h V}$, where $\mathbf{V}$ is some orthogonal matrix. If we define $\mathbf{R} = (\mathbf{I} + (\det \mathbf{V} - 1)\mathbf{hh}^\top)\mathbf{V}$, it is easy to confirm that this is a rotation. Since $\mathbf{P_h h} = \mathbf{0}$, we see that

$$\mathbf{P_h R} = (\mathbf{P_h} + (\det \mathbf{V} - 1)\mathbf{P_h hh}^\top)\mathbf{V} = \mathbf{P_h V} = \mathbf{T}. \quad (27)$$

Thus, $\mathbf{T}$ is a degenerate rotation. □

*Corollary 1:* If $\mathbf{T}$ is a degenerate rotation, then the projection $\mathbf{P_h}$ and the rotation $\mathbf{R}$ such that $\mathbf{T} = \mathbf{P_h R}$ are uniquely determined.

*Proof:* If $\mathbf{T}$ is a degenerate rotation, it is written as $\mathbf{T} = \mathbf{P_h R}$ for some axis $\mathbf{h}$ and rotation $\mathbf{R}$. Matrix $\mathbf{P_h}$ is symmetric with eigenvalues 1, 1, and 0, and hence positive-semi definite, while $\mathbf{R}$ is orthogonal. This means that $\mathbf{T} = \mathbf{P_h R}$ is the polar decomposition of $\mathbf{T}$. This decomposition is unique: Since $\mathrm{rank}\mathbf{P_h} = 2$ and $\det \mathbf{R} = 1$, the rotation $\mathbf{R}$ is the unique solution that maximizes $\mathrm{tr}(\mathbf{R}^\top\mathbf{T})$ (Theorem 2). □

*Corollary 2:* If $\mathbf{T}$ is a degenerate rotation, so is $-\mathbf{T}$. If $\mathbf{T} = \mathbf{P_h R}$ for a rotation $\mathbf{R}$ and a unit vector $\mathbf{h}$, then $-\mathbf{T} = \mathbf{P_h}(\mathbf{I_h R})$, where $\mathbf{I_h} = 2\mathbf{hh}^\top - \mathbf{I}$ is a half-rotation about axis $\mathbf{h}$.

*Proof:* The singular values of $-\mathbf{T}$ are also the singular values of $\mathbf{T}$, which are 1, 1, and 0. Hence, $-\mathbf{T}$ is also a degenerate rotation. Changing the sign of $\mathbf{T} = (\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3)$ means changing the sign of each $\mathbf{t}_i$. Since $\{\mathbf{t}_i\}$ are coplanar, changing their signs means rotating them by angle $\pi$ in the plane on which they lie. Hence, the vectors $\mathbf{r}_i$ to be reconstructed are also rotated by angle $\pi$ around the unit normal $\mathbf{h}$ to the plane. It is easy to see that $\mathbf{I_h}$ is a half-rotation about $\mathbf{h}$, and the resulting rotation is $(\mathbf{I_h r}_1, \mathbf{I_h r}_2, \mathbf{I_h r}_3) = \mathbf{I_h}(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) = \mathbf{I_h R}$. □

Let us call a pair $\{\mathbf{R}, \mathbf{I_h R}\}$ of rotations a *twisted pair*. It is easy to test if a given matrix $\mathbf{T}$ is a degenerate rotation because of Proposition 1.

*Proposition 1:* A matrix $\mathbf{T}$ is a degenerate rotation if and only if

$$\det \mathbf{T} = 0, \|\mathbf{T}\| = \|\mathbf{TT}^\top\| = \sqrt{2}. \quad (28)$$

*Proof:* Let $\lambda_1 \geq \lambda_2 \geq \lambda_3$ ($\geq 0$) be the eigenvalues of symmetric matrix $\mathbf{TT}^\top$. Then,

$$\det(\mathbf{TT}^\top) = \lambda_1\lambda_2\lambda_3, \mathrm{tr}(\mathbf{TT}^\top) = \lambda_1 + \lambda_2 + \lambda_3,$$

$$\mathrm{tr}((\mathbf{TT}^\top)^2) = \lambda_1{}^2 + \lambda_2{}^2 + \lambda_3{}^2. \quad (29)$$

If $\mathbf{T}$ has singular value 1, 1, and 0, then $\lambda_1 = \lambda_2 = 1^2$ and $\lambda_3 = 0^2$. Hence,

$$\det \mathbf{T} = \sqrt{\det(\mathbf{TT}^\top)} = 0, \|\mathbf{T}\| = \sqrt{\mathrm{tr}(\mathbf{TT}^\top)} = \sqrt{2},$$

$$\|\mathbf{TT}^\top\| = \sqrt{\mathrm{tr}((\mathbf{TT}^\top)^\top(\mathbf{TT}^\top))} = \sqrt{\mathrm{tr}((\mathbf{TT}^\top)^2)} = \sqrt{2}. \quad (30)$$

Conversely, suppose (28) holds. Since $(\det \mathbf{T})^2 = \det(\mathbf{TT}^\top) = 0$, it follow from (29) that $\lambda_3 = 0$, and the remaining eigenvalues of $\mathbf{TT}^\top$ satisfy

$$\lambda_1 + \lambda_2 = \lambda_1{}^2 + \lambda_2{}^2 = 2. \quad (31)$$

This means that $\lambda_1 = \lambda_2 = 1$. Hence, matrix $\mathbf{T}$ has singular values $\sqrt{1}$, $\sqrt{1}$, and $\sqrt{0}$. □

In order to obtain a robust method of solving Problem 3, we replace it by

*Problem 4:* Given three arbitrary vectors $\{\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3\}$, find an axis $\mathbf{h}$ such that

$$\sum_{i=1}^{3}(\mathbf{t}_i, \mathbf{h})^2 \to \min, \quad (32)$$

and then find a right-handed orthonormal system $\{\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3\}$ such that

$$\sum_{i=1}^{3}\|\mathbf{P_h r}_i - \mathbf{t}_i\|^2 \to \min. \quad (33)$$

Let us call the above procedure the *optimal resolution* of matrix $\mathbf{T} = (\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3)$ into an axis $\mathbf{h}$ and a rotation $\mathbf{R} = (\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3)$. It appears that we must first compute the axis $\mathbf{h}$ by (32) and then compute (33) from the computed axis $\mathbf{h}$. However, these two steps can be carried out independently as follows:

*optimal-resolution*$(\mathbf{T})$

1) Compute the unit eigenvector $\mathbf{h}$ of matrix $\mathbf{TT}^\top$ for the smallest eigenvalue.

2) Compute the rotation matrix determined by

$$\mathrm{tr}(\mathbf{R}^\top \mathbf{T}) \to \max. \tag{34}$$

*Derivation:* The left-hand side of (32) is written as

$$\sum_{i=1}^{3}(\mathbf{t}_i, \mathbf{h})^2 = \left(\mathbf{h}, \left(\sum_{i=1}^{3}\mathbf{t}_i\mathbf{t}_i^\top\right)\mathbf{h}\right) = (\mathbf{h}, \mathbf{T}\mathbf{T}^\top\mathbf{h}) \to \min. \tag{35}$$

Hence, the axis $\mathbf{h}$ is given by the unit eigenvector (up to sign) of matrix $\mathbf{T}\mathbf{T}^\top$ for the smallest eigenvalue. Take unit eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$ for the remaining eigenvalues so that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{h}\}$ form an orthonormal system. Now,

$$\sum_{i=1}^{3}\|\mathbf{P}_\mathbf{h}\mathbf{r}_i - \mathbf{t}_i\|^2 = \mathrm{tr}(\mathbf{P}_\mathbf{h}^2) - 2\mathrm{tr}((\mathbf{P}_\mathbf{h}\mathbf{R})^\top\mathbf{T}) + \mathrm{tr}(\mathbf{T}^\top\mathbf{T}). \tag{36}$$

Since $\mathrm{tr}((\mathbf{P}_\mathbf{h}\mathbf{R})^\top\mathbf{T}) = \mathrm{tr}(\mathbf{R}^\top\mathbf{P}_\mathbf{h}^\top\mathbf{T}) = \mathrm{tr}(\mathbf{R}^\top\mathbf{P}_\mathbf{h}\mathbf{T})$, the minimization (33) is equivalent to finding a rotation $\mathbf{R}$ such that

$$\mathrm{tr}(\mathbf{R}^\top\mathbf{P}_\mathbf{h}\mathbf{T}) \to \max. \tag{37}$$

The singular value decomposition of $\mathbf{T}$ in vector form is

$$\mathbf{T} = \sigma_1\mathbf{v}_1\mathbf{u}_1^\top + \sigma_2\mathbf{v}_2\mathbf{u}_2^\top + \sigma_3\mathbf{h}\mathbf{u}_3^\top, \tag{38}$$

where $\sigma_1 \geq \sigma_2 \geq \sigma_3$ ($\geq 0$) are the singular values of $\mathbf{T}$, and $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ are an orthonormal system of the unit eigenvectors of $\mathbf{T}^\top\mathbf{T}$. Since $\mathbf{v}_1$ and $\mathbf{v}_2$ are both orthogonal to $\mathbf{h}$, we see that

$$\mathbf{P}_\mathbf{h}\mathbf{T} = \sigma_1\mathbf{v}_1\mathbf{u}_1^\top + \sigma_2\mathbf{v}_2\mathbf{u}_2^\top = \sigma_1\mathbf{v}_1\mathbf{u}_1^\top + \sigma_2\mathbf{v}_2\mathbf{u}_2^\top + 0\mathbf{h}\mathbf{u}_3^\top. \tag{39}$$

This means that $\mathbf{T}$ and $\mathbf{P}_\mathbf{h}\mathbf{T}$ have the same singular value decomposition except for their singular values. If the method of singular value decomposition is applied to solve (37), the singular values themselves do not affect the solution (in fact, the solution is $\mathbf{R} = \mathbf{v}_1\mathbf{u}_1^\top + \mathbf{v}_2\mathbf{u}_2^\top \pm \mathbf{h}\mathbf{u}^\top$, where an appropriate sign is chosen so that $\det\mathbf{R} = 1$ (cf. Theorem 1). Hence, (37) can be replaced by (34). □

## IV. 3-D Motion Estimation

Suppose the camera is rotated by $\mathbf{R}$ around the center of the lens and translated by $\mathbf{h}$. If two images of the same scene before and after the motion are given and point-to-point correspondences of multiple feature points are detected over the two images, we can compute the *motion parameters* $\{\mathbf{R}, \mathbf{h}\}$ and the depth of each of the feature points up to scale [7], [10], [12], [14]. In order to remove the scale indeterminacy, it is customary to scale the translation $\mathbf{h}$ to a unit vector. The core of the problem is summarized as follows:

Let $\mathbf{r}_1$, $\mathbf{r}_2$, and $\mathbf{r}_3$ be the first, second, and third columns of $\mathbf{R}$, respectively. Define the matrix

$$\mathbf{G} = (\mathbf{h}\times\mathbf{r}_1, \mathbf{h}\times\mathbf{r}_2, \mathbf{h}\times\mathbf{r}_3). \tag{40}$$

We abbreviate this matrix as $\mathbf{h}\times\mathbf{R}$. This matrix is called the *essential matrix* and is directly determined from at least eight point-to-point correspondences over the two images. Hence, the problem reduces to the following form:

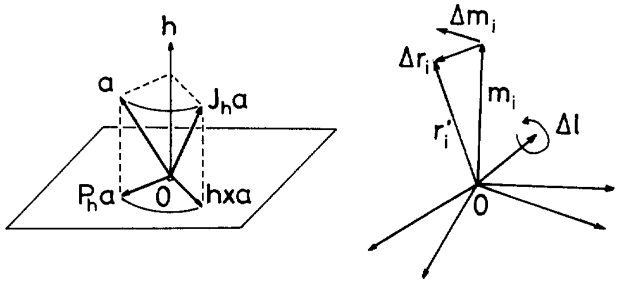
*Problem 5:* For a given matrix $\mathbf{G}$, compute a unit vector $\mathbf{h}$ and a rotation $\mathbf{R}$ such that

$$\mathbf{G} = \mathbf{h}\times\mathbf{R}. \tag{41}$$

Let $\mathbf{J}_\mathbf{h}$ be a quarter-rotation (rotation by angle $\pi/2$) about unit vector $\mathbf{h}$. We observe the following fact:

*Proposition 2:*

$$\mathbf{G} = \mathbf{P}_\mathbf{h}(\mathbf{J}_\mathbf{h}\mathbf{R}). \tag{42}$$




(a) (b)

Fig. 2. (a) Vector product and a quarter-rotation. (b) Perturbation of an orthonormal frame.

*Proof:* If a vector $\mathbf{a}$ is decomposed into the sum $\mathbf{a} = \mathbf{a}_1 + \mathbf{a}_2$, where $\mathbf{a}_1 = (\mathbf{h}, \mathbf{a})\mathbf{h}$ is parallel to unit vector $\mathbf{h}$ and $\mathbf{a}_2 = \mathbf{P}_\mathbf{h}\mathbf{a}$ is perpendicular to $\mathbf{h}$, then $\mathbf{h}\times\mathbf{a}_1 = \mathbf{0}$ and $\mathbf{h}\times\mathbf{a}_2 = \mathbf{J}_\mathbf{h}\mathbf{a}_2$ (see Fig. 2(a)). Hence,

$$\mathbf{h}\times\mathbf{a} = \mathbf{J}_\mathbf{h}\mathbf{a}_2 = \mathbf{J}_\mathbf{h}\mathbf{P}_\mathbf{h}\mathbf{a} = \mathbf{P}_\mathbf{h}\mathbf{J}_\mathbf{h}\mathbf{a}. \tag{43}$$

Since this holds for an arbitrary vector, we obtain the operator identity $\mathbf{h}\times = \mathbf{P}_\mathbf{h}\mathbf{J}_\mathbf{h}$. Applying this to each column of $\mathbf{R}$, we obtain (42).□

Problem 5 is now restated as follows.

*Problem 6:* For a given matrix $\mathbf{G}$, compute a unit vector $\mathbf{h}$ and a rotation $\tilde{\mathbf{R}}$ such that

$$\mathbf{G} = \mathbf{P}_\mathbf{h}\tilde{\mathbf{R}} \tag{44}$$

and then compute

$$\mathbf{R} = \mathbf{J}_\mathbf{h}^\top\tilde{\mathbf{R}}. \tag{45}$$

The first step is robustly computed by the optimal resolution (Problem 4). The optimization proposed by Weng *et al.* [14] is essentially the optimal resolution in our terminology.

We say that a matrix is *decomposable* if it is decomposed into a unit vector and a rotation matrix in the form of (41). The following characterization was first proved by Huang and Faugeras [4], but it is merely a restatement of Theorem 4:

*Proposition 3:* A matrix $\mathbf{G}$ is decomposable if and only if its singular values are 1, 1, and 0.

*Corollary 3:* A matrix $\mathbf{G}$ is decomposable if and only if

$$\det\mathbf{G} = 0, \quad \|\mathbf{G}\| = \|\mathbf{G}\mathbf{G}^\top\| = \sqrt{2}. \tag{46}$$

*Proposition 4:* If matrix $\mathbf{G}$ is decomposable, it can be decomposed in exactly two ways. If $\{\mathbf{R}, \mathbf{h}\}$ and $\{\mathbf{R}', \mathbf{h}'\}$ are the two decompositions, then

$$\mathbf{h}' = -\mathbf{h}, \quad \mathbf{R}' = \mathbf{I}_\mathbf{h}\mathbf{R}. \tag{47}$$

Proposition 4 corresponds to the well-known fact that the true and the spurious rotations form a twisted pair [11].

## V. Statistics of Rotation Fitting

Let $\mathbf{R} = (\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3)$ be a rotation matrix. The three columns $\{\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3\}$ form an orthonormal system. If $\mathbf{R}$ is computed from image data, it may be perturbed into $\mathbf{R}' = (\mathbf{r}_1', \mathbf{r}_2', \mathbf{r}_3')$. However, the three columns $\{\mathbf{r}_1', \mathbf{r}_2', \mathbf{r}_3'\}$ form an orthonormal system, so the error in each element cannot be independent. Since the transformation from $\mathbf{R}$ to $\mathbf{R}'$ is a rotation, the error is also a rotation of some angle $\Delta\Omega$

around some axis l. As is well known (e.g., [5]), if we define $\Delta \mathbf{l} = \Delta\Omega \mathbf{l}$, we have

$$\mathbf{r}_i' = \mathbf{r}_i + \Delta\mathbf{l} \times \mathbf{r}_i + O(\Delta\mathbf{l})^2, \quad i = 1,2,3, \tag{48}$$

where $O(\Delta\mathbf{l})^2$ denotes a term of order 2 or higher in the components of $\Delta\mathbf{l}$. In matrix form,

$$\mathbf{R}' = \mathbf{R} + \Delta\mathbf{l} \times \mathbf{R} + O(\Delta\mathbf{l})^2. \tag{49}$$

Regarding $\Delta\mathbf{l}$ as a vector random variable, we define the *covariance matrix* of rotation $\mathbf{R}$ by

$$V[\mathbf{R}] = E[\Delta\mathbf{l}\Delta\mathbf{l}^\top], \tag{50}$$

where $E[\cdot]$ denotes expectation. The eigenvector of $V[\mathbf{R}]$ for the largest eigenvalue indicates the axis of the error rotation that is most likely to occur, and the corresponding eigenvalue indicates the mean square of the angle of rotation around it.

Consider the problem of fitting an orthonormal frame $\{\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3\}$ to three unit vectors $\{\mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3\}$ by the least-squares criterion

$$\sum_{i=1}^{3} W_i \|\mathbf{r}_i - \mathbf{m}_i\|^2 \to \min. \tag{51}$$

Here, $W_i$ is a weight for the $i$th datum. If we define matrix $\mathbf{R} = (\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3)$, the condition that $\{\mathbf{r}_i\}$, $i = 1, 2, 3$, be a right-handed orthonormal system is equivalent to $\mathbf{R}$ being a rotation matrix. Also define matrix $\mathbf{M} = (W_1\mathbf{m}_1, W_2\mathbf{m}_2, W_3\mathbf{m}_3)$, which can be viewed as the correlation matrix $\sum_{i=1}^{3} W_i \mathbf{m}_i \mathbf{e}_i^\top$, where $\mathbf{e}_i$ is the $i$th coordinate basis vector. Then, it is easy to see that (51) is rewritten as

$$\mathrm{tr}(\mathbf{R}^\top \mathbf{M}) \to \max. \tag{52}$$

Let us call the rotation fitted in this way the *best fitting rotation*. If $\{\mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3\}$ is an orthonormal system from the beginning, we obviously obtain $\mathbf{r}_i = \mathbf{m}_i$, $i = 1,2,3$. In the presence of noise, each $\mathbf{m}_i$ is perturbed by $\Delta\mathbf{m}_i$ (see Fig. 2(b)). Regarding $\Delta\mathbf{m}_i$ as a vector random variable, we define the *covariance matrix* [8] of unit vector $\mathbf{m}_i$ by

$$V[\mathbf{m}_i] = E[\Delta\mathbf{m}_i \Delta\mathbf{m}_i^\top]. \tag{53}$$

The weight $W_i$ should be large for reliable data and small for unreliable data. A reasonable choice is to weigh the term $\|\mathbf{r}_i - \mathbf{m}_i\|$ by the inverse of the root-mean-square error $\sqrt{E[\|\Delta\mathbf{m}_i\|^2]}$ of possible error $\Delta\mathbf{m}_i$. This is equivalent to choosing

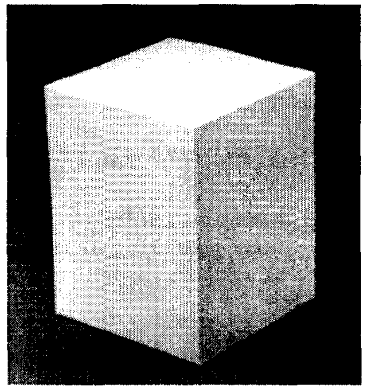
$$W_i = \frac{\mathrm{constant}}{\mathrm{tr}V[\mathbf{m}_i]}. \tag{54}$$

We assume that no two of $W_1$, $W_2$, and $W_3$ are simultaneously 0 and adjust the constant so that $\sum_{i=1}^{3} W_i = 1$.

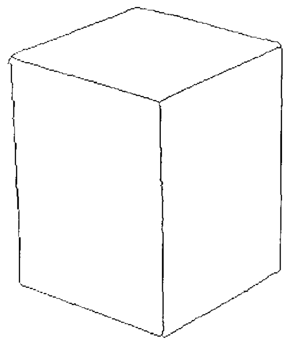
If each $\mathbf{m}_i$ is perturbed by noise into $\mathbf{m}_i' = \mathbf{m}_i + \Delta\mathbf{m}_i$ independently, the fitted vectors $\mathbf{r}_i$ also change into $\mathbf{r}_i' = \mathbf{r}_i + \Delta\mathbf{r}_i$. However, $\{\mathbf{m}_i'\}$ are not necessarily an orthonormal system, and hence each $\Delta\mathbf{r}_i$ is not necessarily equal to each $\Delta\mathbf{m}_i$ (see Fig. 2(b)). Let $|\mathbf{a}, \mathbf{b}, \mathbf{c}|$ (= $(\mathbf{a} \times \mathbf{b}, \mathbf{c}) = (\mathbf{b} \times \mathbf{c}, \mathbf{a}) = (\mathbf{c} \times \mathbf{a}, \mathbf{b})$) be the scalar triple product of vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$. The perturbation of $\mathbf{R}$ is determined as follows:

*Lemma 3:* A perturbation $\Delta\mathbf{m}_i$ of $\mathbf{m}_i$, $i = 1, 2, 3$, causes the best fitting rotation $\mathbf{R}$ to undergo a perturbation by

$$\Delta\mathbf{l} = \sum_{i=1}^{3} \left( \frac{\sum_{j=1}^{3} W_j |\mathbf{r}_i, \mathbf{r}_j, \Delta\mathbf{m}_j|}{1 - W_i} \right) \mathbf{r}_i. \tag{55}$$



(a)



(b)

Fig. 3. (a) A real image of a rectangular box. (b) Detected edges.

*Proof:* According to (51), $\Delta\mathbf{r}_i$ are determined so that

$$\begin{aligned} J &= \sum_{i=1}^{3} W_i \|\Delta\mathbf{r}_i - \Delta\mathbf{m}_i\|^2 \\ &= \sum_{i=1}^{3} W_i \left(\|\Delta\mathbf{r}_i\|^2 - 2(\Delta\mathbf{r}_i, \Delta\mathbf{m}_i) + \|\Delta\mathbf{m}\|^2\right) \to \min. \end{aligned} \tag{56}$$

From (48), we can write $J$ in terms of $\Delta\mathbf{l}$ as

$$J = \sum_{i=1}^{3} W_i \left(\|\Delta\mathbf{l} \times \mathbf{r}_i\|^2 - 2|\Delta\mathbf{l}, \mathbf{r}_i, \Delta\mathbf{m}_i| + \|\Delta\mathbf{m}\|^2\right). \tag{57}$$

If $\Delta\mathbf{l}$ attains the minimum of $J$, an arbitrary perturbation $\Delta\mathbf{l} \to \Delta\mathbf{l} + \delta\Delta\mathbf{l}$ causes zero first variation of $J$ in $\delta\Delta\mathbf{l}$. To a first approximation in $\delta\Delta\mathbf{l}$,

$$\delta J = 2(\delta\Delta\mathbf{l}, \sum_{i=1}^{3} W_i(\Delta\mathbf{l} - (\mathbf{r}_i, \Delta\mathbf{l})\mathbf{r}_i - \mathbf{r}_i \times \Delta\mathbf{m}_i)), \tag{58}$$

which must vanish for arbitrary $\delta\Delta\mathbf{l}$. Hence,

$$\sum_{i=1}^{3} W_i(\Delta\mathbf{l} - (\mathbf{r}_i, \Delta\mathbf{l})\mathbf{r}_i) = \sum_{i=1}^{3} W_i \mathbf{r}_i \times \Delta\mathbf{m}_i. \tag{59}$$

If we define matrix

$$\mathbf{L} = \sum_{i=1}^{3} W_i(\mathbf{I} - \mathbf{r}_i\mathbf{r}_i^\top), \tag{60}$$

the left-hand side of (59) is written as $\mathbf{L}\Delta\mathbf{l}$. Since $\sum_{i=1}^{3} \mathbf{r}_i\mathbf{r}_i^\top = \mathbf{I}$ and $\sum_{i=1}^{3} W_i = 1$, we can write

$$\mathbf{L} = \sum_{i=1}^{3} W_i \sum_{j=1}^{3} \mathbf{r}_j\mathbf{r}_j^\top - \sum_{i=1}^{3} W_i\mathbf{r}_i\mathbf{r}_i^\top = \sum_{i=1}^{3}(1-W_i)\mathbf{r}_i\mathbf{r}_i^\top. \quad (61)$$

Its inverse is given by

$$\mathbf{L}^{-1} = \sum_{i=1}^{3} \frac{\mathbf{r}_i\mathbf{r}_i^\top}{1-W_i}. \quad (62)$$

Since no two of $\{W_i\}$ are simultaneously 0, we have $W_i \neq 1$. From 59), we obtain

$$\Delta\mathbf{l} = \sum_{i,j=1}^{3} \frac{W_j\mathbf{r}_i(\mathbf{r}_i, \mathbf{r}_j \times \Delta\mathbf{m}_i)}{1-W_i}, \quad (63)$$

which is rewritten as (55). □

From this, it is easy to see that the covariance matrix $V[\mathbf{R}] = E[\Delta\mathbf{l}\Delta\mathbf{l}^\top]$ is given as follows:

*Theorem 5:* If each $\mathbf{m}_i$ is independent and has covariance matrix $V[\mathbf{m}_i]$, the covariance matrix $V[\mathbf{R}]$ of the best fitting rotation $\mathbf{R}$ to $\{\mathbf{m}_i\}$, $i = 1, 2, 3$, is given by

$$V[\mathbf{R}] = \sum_{i,j=1}^{3} \frac{\sum_{k=1}^{3} W_k{}^2(\mathbf{r}_i \times \mathbf{r}_k, V[\mathbf{m}_k](\mathbf{r}_j \times \mathbf{r}_k))}{(1-W_i)(1-W_j)}\mathbf{r}_i\mathbf{r}_j^\top. \quad (64)$$

*Example:* Fig. 3(a) is a real image (270 × 300 pixels) of a rectangular box, and Fig. 3(b) shows detected edges. The focal length is estimated to be $f = 1750$ (pixels). The unit vectors that point to the three vanishing points are estimated by least squares as follows:

$$\mathbf{m}_1 = \begin{pmatrix} 0.244 \\ -0.792 \\ 0.559 \end{pmatrix}, \quad \mathbf{m}_2 = \begin{pmatrix} 0.300 \\ 0.636 \\ 0.711 \end{pmatrix}, \quad \mathbf{m}_3 = \begin{pmatrix} -0.914 \\ 0.030 \\ 0.405 \end{pmatrix}. \quad (65)$$

(The $X$-axis extends upward, the $Y$-axis rightward, and the $Z$-axis away from the viewer.) These indicate the three 3-D orientations of the edges [6]. If the image resolution $\kappa$ is assumed to be unity, their covariance matrixes can be evaluated theoretically as follows [8]:

$$V[\mathbf{m}_1] = 10^{-5}\begin{pmatrix} 0.402 & -1.114 & -1.755 \\ -1.114 & 3.790 & 5.857 \\ -1.755 & 5.857 & 9.070 \end{pmatrix},$$

$$V[\mathbf{m}_2] = 10^{-5}\begin{pmatrix} 0.887 & 1.681 & -1.878 \\ 1.681 & 3.505 & -3.845 \\ -1.878 & -3.845 & 4.231 \end{pmatrix},$$

$$V[\mathbf{m}_3] = 10^{-5}\begin{pmatrix} 1.241 & 0.019 & 2.794 \\ 0.019 & 0.023 & 0.041 \\ 2.794 & 0.041 & 6.292 \end{pmatrix}. \quad (66)$$

The best fitting rotation matrix is

$$\mathbf{R} = \begin{pmatrix} 0.239 & 0.320 & -0.917 \\ -0.780 & 0.626 & 0.015 \\ 0.578 & 0.712 & 0.399 \end{pmatrix}. \quad (67)$$

The discrepancies of $\mathbf{m}_1$, $\mathbf{m}_2$, and $\mathbf{m}_3$ from the corresponding orientations are 1.35°, 1.25°, and 0.97°, respectively. By evaluating the covariance matrix $V[\mathbf{R}]$ given by (64), we can see that the root-mean-square error $\Delta\Omega$ of the angle of error rotation (from the true frame, which we do not know) is 0.49°.

## VI. Concluding Remarks

In this paper, we first recapitulated methods of fitting a 3-D rotation to 3-D data in a refined form as optimization over proper rotations, extending three existing methods—the method of singular value decomposition, the method of polar decomposition, and the method of quaternion representation. As an application of these three methods, we formulated the problem of *optimal resolution* of a *degenerate rotation* and showed how this solves the problem of 3-D motion estimation from two images in a succinct way. Finally, we defined the covariance matrix of rotation fitting and analyzed the statistical behavior of error of the fit in terms of the covariance matrixes of the data.

## References


[1] K. S. Arun, T. S. Huang, and S. D. Blostein, "Least-squares fitting of two 3-D point sets," *IEEE Trans. Pattern Anal. Machine Intell.*, vol. 9, pp. 698–700, 1987.

[2] B. K. P. Horn, "Closed-form solution of absolute orientation using unit quaternions," *J. Opt. Soc. Amer.*, vol. A-4, pp. 629–642, 1987.

[3] B. K. P. Horn, H. M. Hilden, and S. Negahdaripour, "Closed-form solution of absolute orientation using orthonormal matrices," *J. Opt. Soc. Amer.*, vol. A-5, pp. 1128–1135, 1988.

[4] T. S. Huang and O. D. Faugeras, "Some properties of the $E$ matrix in two-view motion estimation," *IEEE Trans. Pattern Anal. Machine Intell.*, vol. 11, pp. 1310–1312, 1989.

[5] K. Kanatani, *Group-Theoretical Methods in Image Understanding*, Springer, Berlin, 1990.

[6] K. Kanatani, "Computational projective geometry," *Comput. Vision, Graphics, Image Processing: Image Understanding*, vol. 54, pp. 333–448, 1991.

[7] K. Kanatani, "Unbiased estimation and statistical analysis of 3-D rigid motion from two views," *IEEE Trans. Pattern Anal. Machine Intell.*, vol. 15, pp. 37–50, 1993.

[8] K. Kanatani, "Statistical analysis of geometric computation," *Comput. Vision, Graphics, Image Processing: Image Understanding*, vol. 59, no. 3, 1994.

[9] K. Kanatani, *Geometric Computation for Machine Vision*. Oxford, UK: Oxford University Press, 1993.

[10] H. C. Longuet-Higgins, "A computer algorithm for reconstructing a scene from two projections," *Nature*, vol. 293, no. 10, pp. 133–135, 1981.

[11] H. C. Longuet-Higgins, "Multiple interpretations of a pair of images of a surface," *Proc. Roy. Soc. London*, vol. A-418, pp. 1–15, 1988.

[12] R. Y. Tsai and T. S. Huang, "Uniqueness and estimation of three-dimensional motion parameters of rigid objects with curved surfaces," *IEEE Trans. Pattern Anal. Machine Intell.*, vol. 6, pp. 13–27, 1984.

[13] S. Umeyama, "Least-squares estimation of transformation parameters between two point patterns," *IEEE Trans. Pattern Anal. Machine Intell.*, vol. 13, pp. 376–380, 1991.

[14] J. Weng, T. S. Huang, and N. Ahuja, "Motion and structure from two perspective views: Algorithms, error analysis, and error estimation," *IEEE Trans. Pattern Anal. Machine Intell.*, vol. 11, pp. 451–467, 1989.