Short Papers

Comments on "Symmetry as a Continuous Feature"

Kenichi Kanatani, *Senior Member, IEEE*

Abstract—We point out the existence of a theoretical difficulty that underlies the symmetry detection studied by Zabrodsky et al. [4] and present a possible solution to it.

Index Terms—Symmetry detection, shape classification, pattern recognition, geometric AIC, statistical analysis.

1 Introduction

THE goal of Zabrodsky et al. [4] is to detect a symmetry in a figure extracted from an image. Their basic strategy is to choose the symmetry that is the "closest" to the figure measured by an appropriate metric, as which they adopted the minimum sum of the squared distances over which the vertices must be moved to impose the assumed symmetry; they called it the *symmetry distance*. It follows that we need an algorithm for efficiently imposing a given symmetry with a minimum displacement. The paper of Zabrodsky et al. [4] is almost entirely devoted to such algorithms; little attention is paid to the original goal of symmetry detection. Here, we point out the existence of a theoretical difficulty that underlies this problem and present a possible solution to it.

2 Hierarchy of Symmetries

The problem lies in the fact that symmetries have *hierarchy*: C_8 symmetry implies C_4 symmetry, which implies C_2 symmetry, for example. As a result, classes of figures that have a particular symmetry have *inclusion relations* among themselves: The figures with C_8 symmetry is a subset of the figures with C_4 symmetry, which is a subset of the figures with C_2 symmetry.

For simplicity, let us restrict the figures to polygons with a fixed number n of vertices, as Zabrodsky et al. [4] did. Consider the case of $n = 4$, for example. Suppose we want to test if a given quadrangle can be judged to be a square or a rectangle. Since the set of squares is a subset of the set of rectangles, the symmetry distance to a rectangle is always no more than that to a square. Hence, a quadrangle is never judged to be a square. The case of $n = 6$ is shown in Fig. 5 in Zabrodsky et al. [4]. We can see that weaker symmetries indeed have smaller symmetry distances.

Let us state the problem in general terms. The task is to find an optimal class for a given input pattern. This is the goal of *pattern recognition*, and usually the one that is the "closest" to the input measured by an appropriate metric is chosen. However, it is tacitly assumed in pattern recognition that the classes into which the input is to be classified are *disjoint*. If class A is *included* in class B, class A is never chosen because the distance to class A is always not smaller than that to class B (Fig. 1).

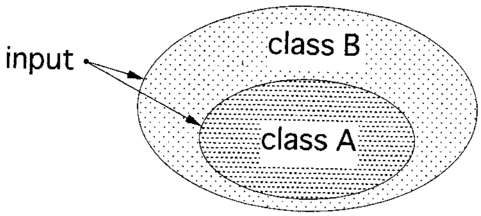

Fig. 1. Class A is never chosen whatever distance measure is used.

3 Class Selection by Geometric AIC

In order to give every class a fair chance, we must introduce another criterion that favors classes that are included in others. A similar problem appears in statistics, and the *AIC* (*Akaike information criterion*) gives one solution [1]. However, the AIC is formulated for a *statistical inference* formalized as estimating the parameter of a statistical distribution from multiple samples chosen from it. As a result, the AIC cannot be applied to *geometric inference* in its original form in general. However, if we go back to the principle that gives rise to the AIC, we can obtain a similar criterion, called the *geometric AIC* [2], [3], applicable to geometric inference in general.

The principle underlying the geometric AIC is the *robustness to perturbation*. As an illustration, consider the case of polygons with n vertices. If C_n symmetry is to be imposed, there exist four degrees of freedom (the position of the center of symmetry and the position of one vertex), with respect to which the symmetry distance is minimized. If $C_{n/2}$ symmetry is to be imposed (assuming that n is even), there exist six degrees of freedom (the position of the center of symmetry and the positions of two consecutive vertices). Thus, each symmetry class has certain degrees of freedom, with respect to which the symmetry measure is minimized, and a class with a stronger symmetry has less degrees of freedom.

Given a polygon s, let a be the best polygon chosen for it from class A. Consider another polygon s', and let a' be the best polygon for it chosen from class A. We omit the precise definition (see [2], [3]), but intuitively, polygon a is "robust"when *s and s' are close, then a is also a good approximation to s'*. It follows that if class A is included in class B, a polygon from class A is always more robust than one from class B. In fact, the set A' of polygons in class A that are within a distance from s is smaller than the set B' of polygons in class B that are within the same distance from s. Hence, the set of polygons that are approximated by the polygons in B' is larger than the set of polygons that are approximated by the polygons in A'.

Using the same statistical model and the same mathematical analysis that Zabrodsky et al. [4] used, i.e., assuming that each vertex is perturbed from its true position by independent isotropic Gaussian noise, adopting *maximum likelihood estimation*, and analyzing the χ^2 distribution of the residual of the symmetry distance, we conclude as follows (see [3] for the details). Let $a \in A$ be the best polygon in class A with f_A degrees of freedom, and $b \in B$ the best polygon in class B with f_B degrees of freedom. Let SD_a and SD_b be

• *K. Kanatani is with the Dept. of Computer Science, Gunma University, Kiryu, Gunma 371, Japan. E-mail: kanatani@cs.gunma-u.ac.jp.*

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their symmetry distances. Suppose class A is included in class B: $A \subset B$. According to the geometric AIC, class A is preferred to class B if

$$\frac{SD_a}{SD_b} < 1 + \frac{2(f_b - f_a)}{2n - f_b}. \tag{1}$$

For $n = 4$, for example, a square is preferred to a rectangle if

$$\frac{SD_{square}}{SD_{rectangle}} < \frac{5}{3}. \tag{2}$$

For $n = 6$, consider the example shown in Fig. 5 of Zabrodsky et al. [4]: $SD_2 = 1.87$, $SD_3 = 1.64$, $SD_6 = 2.53$ (SD_n denotes the symmetry distance for C_n symmetry). Noting that C_6 symmetry is a kind of C_2 symmetry, we conclude that C_6 symmetry is preferred to C_2 symmetry if

$$\frac{SD_6}{SD_2} < 3. \tag{3}$$

Noting that C_6 symmetry is also a kind of C_3 symmetry, we conclude that C_6 symmetry is preferred to C_3 symmetry if

$$\frac{SD_6}{SD_2} < \frac{5}{3}. \tag{4}$$

In the example of Zabrodsky et al. [4], we have $SD_6 / SD_2 = 1.3529...$ and $SD_6 / SD_3 = 1.5426....$ Hence, C_6 symmetry is preferred to both C_2 and C_3; the input figure is judged to have C_6 symmetry.

4 Concluding Remarks

The kind of reasoning stated above is expected to play a crucial role in building an intelligent system for automatically detecting a symmetry in an image.

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