A Cosserat Continuum Model for Vibrating Frameworks

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A complex-valued Cosserat continuum model is constructed for grid frameworks vibrating with an arbitrary frequency by means of a variational principle related to the average energy of the system. The degree of approximation is taken into consideration. Forced vibration is analyzed, and resonance effects, which are not seen in the usual continuous models, are noted. Analysis of wave propagation reveals the existence of high frequency waves. The accuracy of solutions is also investigated.

I. INTRODUCTION

In frame analysis the displacements and the rotations of joints are taken as unknown variables, and equations of equilibrium are solved to determine them. Usually electronic computers are used, but the calculations become expensive, especially when the structure is large. Hence, a suitable technique for approximation is desired. For this purpose continuous approximation has been frequently used; elastic plates and shells have long been used as models for lattice plates and shells. Recently Cosserat, or micropolar continua[1], have come to be used.[2-5] Continuous approximation is especially effective in dynamic problems, because exact calculation is more difficult than in static cases. Sun and Yang[5] showed that the Cosserat continuum model can be employed for dynamic problems with great ease. However, their model cannot describe such an important phenomenon as resonance, because they made an implicit assumption that static characteristics of member beams are valid even when they are in motion. But general equations of motion of an equivalent continuum cannot be obtained without this assumption. In this paper we shall circumvent this difficulty by restricting our attention to vibration. We shall construct a complex-valued Cosserat continuum model for vibrating grid frameworks applicable to vibration of any frequency. Equations of vibration are derived by means of a variational principle related to the average energy of the system. In order to convert a discrete system to an equivalent continuum model, Taylor expansion has commonly been used[2-5] with only the first few terms retained. This procedure, however does not lead to reasonable equation, if the degree of approximation is taken into consideration. In this paper we shall follow a suitable principle of approximation. Forced vibration is analyzed, and resonance effects, which cannot be seen in the model of Sun and Yang,[5] are observed. Wave propagaion is also analyzed, and the existence of high frequency waves, which were also missing in Ref[5], is shown. Sun and Yang showed that their analysis was in good agreement with several sample solutions calculated by the finite element method in the range of low frequency. In this paper we shall show that our results are in fairly good agreement with exact solutions in a wide range of frequency, if the order of continuous approximation is taken properly.

II. EQUATIONS OF VIBRATION OF GRID FRAMEWORKS

Consider a moving beam as shown in Fig.1. The kinetic energy of the beam is

$$\frac{1}{2}\int_0^l \rho A[\dot{x}(s)^2 + \dot{y}(s)^2 + \dot{z}(s)^2]ds., \tag{1}$$

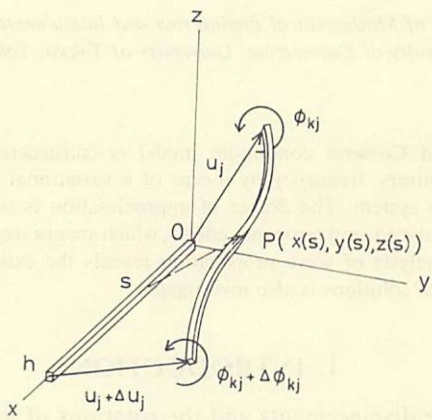

Fig. 1. Deformed and undeformed configurations of a member beam.

where ρA is the line density, and $\cdot$ denotes d/dt. If there exists an equivalent continuum, then (1) must be expressed only in terms of the variables at both ends. Sun and Yang[5] implicitly assumed static bending and expressed (1) in terms of v_j, Δv_j, ω_{kj} and $\Delta\omega_{kj}$ ($v_j \equiv \dot{u}_j$, $\omega_{kj} \equiv \dot{\phi}_{kj}$). Using Taylor expansions of the relative velocity Δv_j and the relative angular velocity $\Delta\omega_{kj}$, Sun and Yang derived equations of motion by Hamilton's principle. We call their model the *quasistatic* model, for it is valid only for slow motion. But without the assumption of static bending, general equations of motion of an equivalent continuum cannot be obtained, and hence such an important phenomenon as resonance cannot be described. We shall circumvent this difficulty by restricting our attention to vibration alone. Then we can make use of a variational principle related to the average energy of the system to obtain a complex-valued Cosserat continuum model for grid frameworks vibrating with an arbitrary frequency.

Let x^α ($\alpha = 1,2, \ldots ,n$) be the generalized coordinate for a linear mechanical system. Equations of motion have the form

$$m_{\beta\alpha}\,\ddot{x}^\beta + k_{\beta\alpha}\,x^\beta + d_{\beta\alpha}\,\dot{x}^\beta = Q_\alpha, \tag{2}$$

where $m_{\beta\alpha}$, $k_{\beta\alpha}$ and $d_{\beta\alpha}$ are the mass matrix, the stiffness matrix and the damping matrix of the system respectively, and Q_α is the force acting against the generalized coordinate x^α. In the usual analysis of vibration, x^α, $Q_\alpha \propto e^{i\omega t}$ ($i = \sqrt{-1}$) is assumed. Then the equations of vibration of the system are

$$-\omega^2\,m_{\beta\alpha}\,x^\beta + k_{\beta\alpha}\,x^\alpha + i\omega d_{\beta\alpha}\,x^\beta = Q_\alpha. \tag{3}$$

Now x^α and Q_α are complex quantities. We now construct a variational principle which leads directly to the complex expression (3). Consider the time average of the kinetic energy K, the potential energy U, and the dissipation function F. We obtain Hermitian forms

$$\langle K\rangle = \frac{\omega^2}{4} m_{\beta\alpha} x^\alpha \bar{x}^\beta, \quad \langle U\rangle = \frac{1}{4} k_{\beta\alpha} x^\alpha \bar{x}^\beta, \quad \langle F\rangle = \frac{\omega^2}{4} d_{\beta\alpha} x^\alpha \bar{x}^\beta, \tag{4}$$

where $\bar{x}^\alpha$ is the complex conjugate of x^α. By $\langle\ \rangle$ is denoted the time average. Then we can deduce the following principle.

$$\frac{\partial S}{\partial x^\alpha} = Q_\alpha, \qquad S \equiv \frac{1}{4}(\langle U\rangle - \langle K\rangle + (i/\omega)\langle F\rangle). \tag{5}$$

The number n of the variables is irrelevent, so this must also hold in the limit of continuum, in which partial derivatives must be replaced by functional derivatives with respect to the corresponding field variables.

Consider a vibrating beam shown in Fig. 2, where u, Δu, v, Δv, ϕ and $\Delta\phi$ may be complex. If the longitudinal vibration of the beam is neglected, we can put

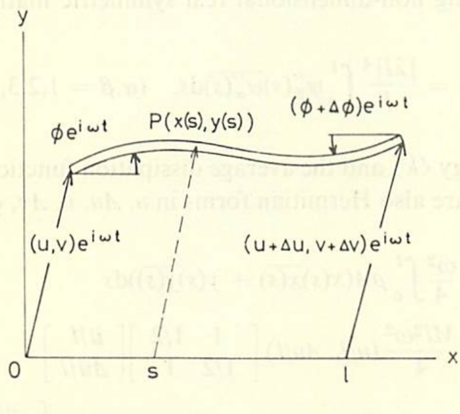

Fig. 2. Vibration of a member beam.

$$x(s) = \left(u + \frac{s}{l}\Delta u\right)e^{i\omega t}, \tag{6}$$

and $y(s)$ is determined by

$$EIy''''(s) + \eta\dot{y}(s) + \rho A\ddot{y}(s) = 0, \tag{7}$$

where η is the damping coefficient. Putting $y \propto e^{i\omega t}$, we have

$$y''''(s) = \lambda^4 y(s), \qquad \lambda \equiv \sqrt[4]{(\rho A\omega^2 - i\omega\eta)/EI}. \tag{8}$$

The boundary conditions are

$$y(0) = ve^{i\omega t}, \qquad y'(0) = \phi e^{i\omega t},$$
$$y(l) = (v + \Delta v)e^{i\omega t}, \qquad y'(l) = (\phi + \Delta\phi)e^{i\omega t}. \tag{9}$$

Then $y(s)$ is determined in the following linear form in v, Δv, ϕ and $\Delta\phi$.

$$y(s) = (v\psi_1(s) + \Delta v\psi_2(s) + l\phi\psi_3(s) + l\Delta\phi\psi_4(s))e^{i\omega t}. \tag{10}$$

Here ψ_α's are complex-valued non-dimensional functions of s. (See Appendix.) If the strain energy due to elongation and twisting is neglected, the average strain energy of bending is

$$\langle\varepsilon\rangle = \frac{1}{4}\int_0^l EIy''(s)\overline{y''(s)}\mathrm{d}s$$
$$= \frac{EI}{4l}(v/l,\ \Delta v/l,\ \phi,\ \Delta\phi)[C_{\beta\alpha}(\lambda l)]\begin{bmatrix}\bar{v}/l\\ \Delta\bar{v}/l\\ \bar{\phi}\\ \Delta\bar{\phi}\end{bmatrix}, \tag{11}$$

where $C_{\beta\alpha}$ is the following non-dimensional real symmetric matrix.

$$C_{\beta\alpha}(\lambda l) = \frac{|\lambda l|^4}{l}\int_0^l \psi''_\beta(s)\overline{\psi''_\alpha(s)}\mathrm{d}s. \quad (\alpha,\beta = 1,2,3,4) \tag{12}$$

The average kinetic energy $\langle k\rangle$ and the average dissipation function $\langle f\rangle$ are determined in the same manner. They are also Hermitian forms in u, Δu, v, Δv, ϕ and $\Delta\phi$.

$$\langle k\rangle = \frac{\omega^2}{4}\int_0^l \rho A(x(s)\overline{x(s)} + y(s)\overline{y(s)})\mathrm{d}s$$
$$= \frac{Ml^2\omega^2}{4}(u/l,\ \Delta u/l)\begin{bmatrix}1 & 1/2\\ 1/2 & 1\end{bmatrix}\begin{bmatrix}\bar{u}/l\\ \Delta\bar{u}/l\end{bmatrix}$$
$$+ \frac{Ml^2\omega^2}{4}(v/l,\ \Delta v/l,\ \phi,\ \Delta\phi)\,[M_{\beta\alpha}(\lambda l)]\begin{bmatrix}\bar{v}/l\\ \Delta\bar{v}/l\\ \bar{\phi}\\ \Delta\bar{\phi}\end{bmatrix}. \tag{13}$$

$$\langle f\rangle = \frac{\omega^2}{4}\int_0^l \eta y(s)\overline{y(s)}\mathrm{d}s$$
$$= \frac{\eta l^3\omega^2}{4}(v/l,\ \Delta v/l,\ \phi,\ \Delta\phi)\,[M_{\beta\alpha}(\lambda l)]\begin{bmatrix}\bar{v}/l\\ \Delta\bar{v}/l\\ \bar{\phi}\\ \Delta\bar{\phi}\end{bmatrix}. \tag{14}$$

$$M_{\alpha\beta}(\lambda l) = \frac{1}{l}\int_0^l \psi_\beta(s)\overline{\psi_\alpha(s)}\mathrm{d}s. \quad (\alpha,\beta = 1,2,3,4) \tag{15}$$

Now consider a three-dimensional grid framework in vibration. Repeating the above procedure, we can obtain the total average kinetic energy $\langle K \rangle$, the total average strain energy $\langle U \rangle$, and the total average dissipation function $\langle F \rangle$ in Hermitian forms. Put

$$S \equiv (\langle U \rangle - \langle K \rangle + (i/\omega)\langle F \rangle)/4, \tag{16}$$

Then S is expressed in the form

$$S = \sum_{I,J,K} s(u_j(I,J,K), \overline{u_j(I,J,K)}, \Delta_k u_j(I.J.K), \Delta_k \overline{u_j(I,J,K)}, \phi_{kj}(I,J,K), \overline{\phi_{kj}(I,J,K)}, \Delta_l \phi_{kj}(I,J,K), \Delta_l \overline{\phi_{kj}(I,J,K)}) \tag{17}$$

where $u_j(I,J,K)$ and $\phi_{kj}(I,J,K)$ are respectively the amplitude of the displacement and the amplitude of the rotation of the joint labeled as (I,J,K), and Δ_j denotes the finite difference operator in the j-direction.[6] An equivalent continuous model is obtained by replacing the finite difference terms by their Taylor expansions

$$\begin{aligned} \Delta_k u_j &= h_k \partial_k u_j + \frac{1}{2} h_k^2 \partial_k^2 u_j + \ldots, \\ \Delta_l \phi_{kj} &= h_l \partial_l \phi_{kj} + \frac{1}{2} h_l^2 \partial_l^2 \phi_{kj} + \ldots, \end{aligned} \tag{18}$$

where h_j is the length of beams lying in the j-direction, and ∂_j denotes $\partial/\partial x^j$. To convert a discrete system to an equivalent continuous model, Taylor expansion is also used in Refs. (2)–(5) where only the first terms are retained. This is, however, not a reasonable approximation here, because s is a Hermitian form. The validity of omitting succeeding terms comes from the fact that they are quantities of higher order smallness. Then the term $h_k \partial_k u_j h_m \partial_m \bar{u}_l$, for example, is of second order smallness and it is comparable with the omitted term (1/2) $h_k \partial_k \bar{u}_j$ multiplied by the 0 th order term $\bar{u}_j$. To avoid this inconsistency, we adopt the following procedure: substitute formally the infinite series (18) into (17) and omit those terms of order of smallenss higher than N. Let us call this approximation the *N-th order continuous approximation*, and that used in Refs. (2)–(5) the *simple approximation*. However, Bazant and Christensen[4] noticed that important terms were missing and added them. A consistent principle is given by our method of approximation.

After our procedure of approximation, we can approximate S by

$$S = \int s(u_j, \bar{u}_j, \partial_k u_j, \partial_k \bar{u}_j, \ldots, \phi_{kj}, \overline{\phi_{kj}}, \ldots) \mathrm{d}V. \tag{19}$$

Taking functional derivatives with respect to the complex conjugate variables according to principle (5), we obtain the following equations of vibration.

$$\begin{aligned} (\delta S/\delta \overline{u_j} =)\ \partial s/\partial \overline{u_j} - \partial_k(\partial s/\partial(\partial_k \overline{u_j})) + \ldots &= b^j, \\ (\delta S/\delta \overline{\phi_{kj}} =)\ \partial s/\partial \overline{\phi_{kj}} - \partial_l(\partial s/\partial(\partial_l \overline{\phi_{kj}})) + \ldots &= 0, \end{aligned} \tag{20}$$

where we have put the force acting on joints per unit volume to be $b^j e^{i\omega t}$ and put the joint moment to be zero.

III. APPLICATIONS AND ACCURACY OF SOLUTIONS

Consider shearing vibration of a two-dimensional grid framework of infinite extent. The grid is assumed to be square with identical members of length l, stiffness EI and line density ρA. We adopt the natural units; we measure all length in terms of l, all masses in terms of ρAl, and the time in terms of $l^2\sqrt{\rho A/EI}$. Let us put $u_j = (0, v(x))e^{i\omega t}$, $\phi = \phi(x)e^{i\omega t}$ and $b^j = (0, b(x))e^{i\omega t}$. In the second approximation, equations (20) are reduced to

$$A_1 v + A_1 v'' + C\phi' = b, \quad -Cv' + B_1\phi + B_2\phi'' = 0, \tag{21}$$

where

$$\begin{aligned} A_1 &= \omega^2(M_{11} + 1) + C_{11} + i\omega\eta M_{11}, \\ A_2 &= -\omega^2(M_{12} - M_{22}) + (C_{12} - C_{22}) + i\omega\eta(M_{12} - M_{22}), \\ B_1 &= -2\omega^2 M_{33} + 2C_{33} + 2i\omega\eta M_{33}, \\ B_2 &= -\omega^2(M_{34} - M_{44}) + (C_{34} - C_{44}) + i\omega\eta(M_{34} - M_{44}), \\ C &= -\omega^2(M_{14} - M_{23}) + (C_{14} - C_{23}) + i\omega\eta(M_{14} - M_{23}). \end{aligned} \tag{22}$$

Consider M_{11}, for example. It plays the role of virtual mass density of the vibrating grid framework and depends on ω and η as shown in Fig.3. The angular frequency $\omega_0 = 22.37\sqrt{EI/\rho A}/l^2$ is the first characteristic angular frequency of member beams. Similarly M_{33} plays the role of virtual moment of inertia density (Fig.4). If all $C_{\beta\alpha}$'s and $M_{\beta\alpha}$'s in (22) are

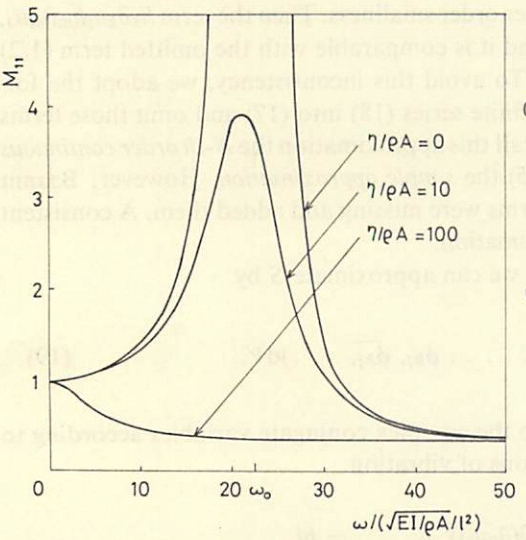

Fig. 3. Virtual mass M_{11} of vibrating member beams. $\omega_0 = 22.37\sqrt{EI/\rho A}/l^2$.

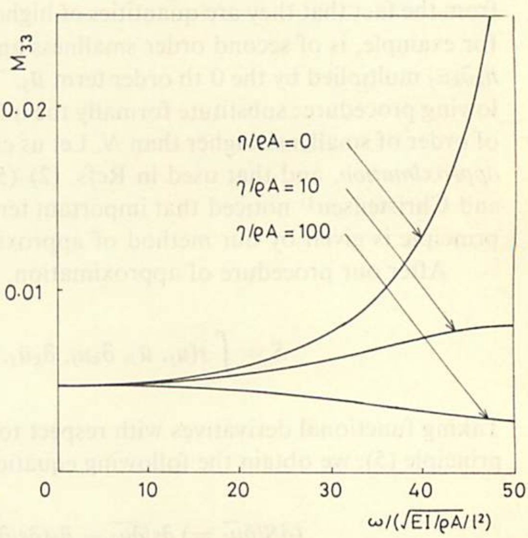

Fig. 4. Virtual moment of inertia M_{33} of vibrating member beams.

replaced by their respective values at $\omega = 0$ (see Appendix), then Eqs. (22) are reduced to those of the quasistatic model.

Let us consider the problem of forced vibration. We put the shearing force to be $b(x) = b_0x$, and seek a solution to Eq. (21) in the form $v(x) = v_0x$ and $\phi(x) = \phi_0$. The magnitudes of response $|v_0/b_0|$ and $|\phi_0/b_0|$ are plotted in Fig. 5 and Fig. 6, respectively versus the angular frequencies. The dashed curves show the response of the quasistatic model, and the resonance effect in our model is missing in the quaistatic model. Since continuous models are obtained by the neglect of higher order derivatives, solutions are fairly accurate for low order modes of vibration. The above forced vibration is of the linear mode, and hence the solutions are exact in this case. If higher order modes are assumed, we must consider the accuracy of the solution. A systematic investigation of this is possible if we study the problem of wave propagation, for the wavelength plays the role of the mode of vibration. Let us assume $v(x)$ and $\phi(x)$ in (21) to be proportional to e^{-ikx}. Then we obtain a set of algebraic equations. Wave propagation is possible only when the determinant vanishes. Figure 7 shows the angular frequency of the wave of wave number k, including those of the first, the third, and the fourth order approximations. Exact solutions* are shown by the thick solid curves in Fig. 7. We can conclude that the second order approximation is accurate enough for $kl \leqq 0.5$, while the third order approximation is necessary to keep enough accuracy for $kl \leqq 1$.** We should note that the high frequency waves shown in Fig. 7 cannot be seen

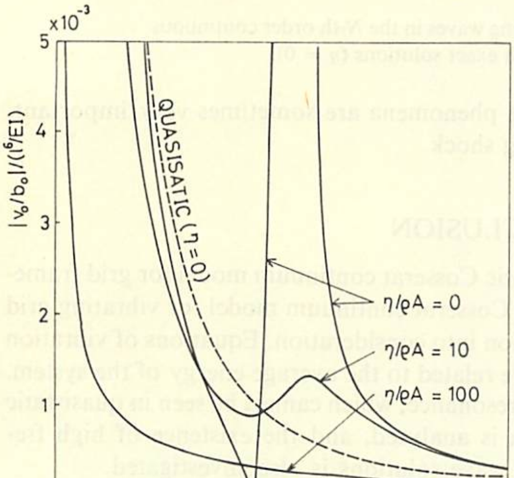

Fig. 5. Response $|v_0/b_0|$ under forced shearing vibration. $\omega_1 = 27.62 \sqrt{EI/\rho A/l^2}$. The dashed curve is the quasistatic solution.

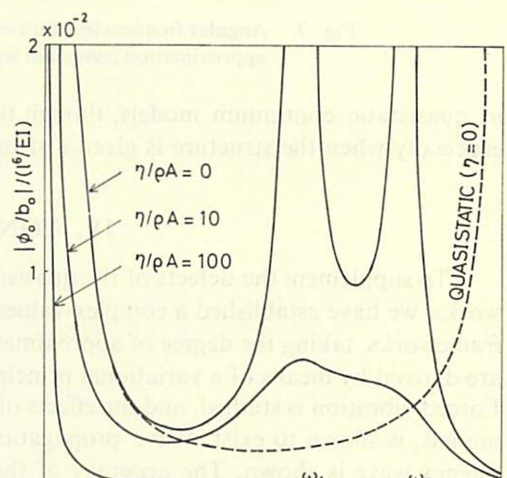

Fig. 6. Response $|\phi_0/b_0|$ under forced shearing vibration. $\omega_2 = 39.48 \sqrt{EI/\rho A/l^2}$. The dashed curve is the quasistatic solution.

*Sun and Yang[5] also calculated the wave frequency by the quasistatic model of simple approximation in the range of low frequency, but they did not give exact solutions. Instead they gave several numerical examples calculated by the finite element method. However, exact solutions can be obtained also by our variational principle of average energy.

**Due to the regularity of the grid, $kl = \pi$ corresponds to the shortest possible wavelength. The situation is similar to the *Brillouin zone* in crystal physics.

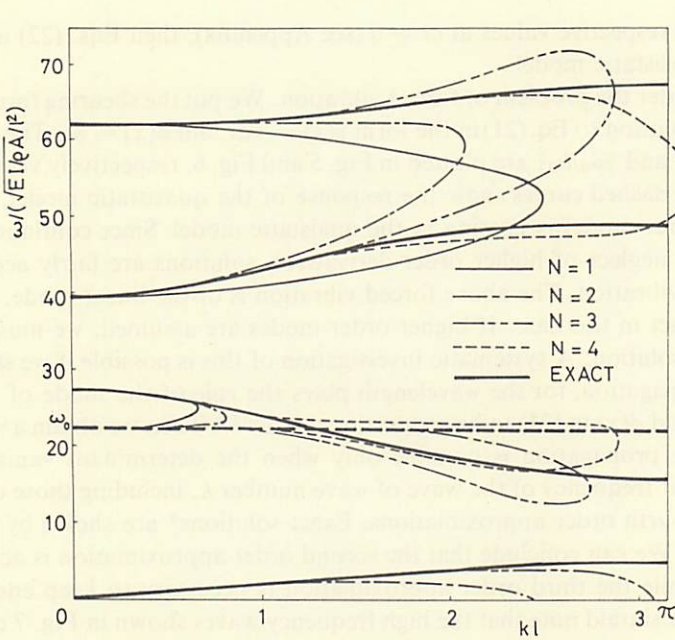

Fig. 7. Angular frequencies of traveling waves in the N-th order continuous approximation compared with exact solutions ($\eta = 0$).

in quasistatic continuum models, though the phenomena are sometimes very important, especially when the structure is given a strong shock.

IV. CONCLUSION

To supplement the defects of the quasistatic Cosserat continuum model for grid frameworks, we have established a complex-valued Cosserat continuum model for vibrating grid frameworks, taking the degree of approximation into consideration. Equations of vibration are derived by means of a variational principle related to the average energy of the system. Forced vibration is studied, and the effects of resonance, which cannot be seen in quasistatic models, is shown to exist. Wave propagation is analyzed, and the existence of high frequency wave is shown. The accuracy of the wave solutions is also investigated.

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APPENDIX

$$\psi_1(s) = \alpha_1(\lambda l)\,(\cos\lambda s - \cosh\lambda s) + \alpha_2(\lambda l)\,(\sin\lambda s - \sinh\lambda s) + \cosh\lambda s,$$

$$\psi_2(s) = \alpha_3(\lambda l)\,(\cos\lambda s - \cosh\lambda s) + \alpha_4(\lambda l)\,(\sin\lambda s - \sinh\lambda s),$$

$$\psi_3(s) = \alpha_5(\lambda l)\,(\cos\lambda s - \cosh\lambda s) + \alpha_6(\lambda l)\,(\sin\lambda s - \sinh\lambda s) + \frac{1}{\lambda l}\sin\lambda s,$$

$$\psi_4(s) = \alpha_7(\lambda l)\,(\cos\lambda s - \cosh\lambda s) + \alpha_8(\lambda l)\,(\sin\lambda s - \sinh\lambda s).$$

$$\alpha_1(\lambda l) = (\cos\lambda l - \cosh\lambda l + \sin\lambda l\sinh\lambda l - \cos\lambda l\cosh\lambda l + 1)/D(\lambda l),$$

$$\alpha_2(\lambda l) = (\sin\lambda l + \sinh\lambda l - \cos\lambda l\sinh\lambda l - \sin\lambda l\cosh\lambda l)/D(\lambda l),$$

$$\alpha_3(\lambda l) = (\cos\lambda l - \cosh\lambda l)/D(\lambda l),$$

$$\alpha_4(\lambda l) = (\sin\lambda l + \sinh\lambda l)/D(\lambda l),$$

$$\alpha_5(\lambda l) = (-\sin\lambda l + \sinh\lambda l + \sinh\lambda l\cosh\lambda l - \cos\lambda l\sin\lambda l)/\lambda l D(\lambda l),$$

$$\alpha_6(\lambda l) = (\cos\lambda l - \cosh\lambda l - \cos\lambda l\cosh\lambda l - \sin\lambda l\sinh\lambda l + 1)/\lambda l D/(\lambda l),$$

$$\alpha_7(\lambda l) = (-\sin\lambda l + \sinh\lambda l)/\lambda l D(\lambda l),$$

$$\alpha_8(\lambda l) = (\cos\lambda l - \cosh\lambda l)/\lambda l D(\lambda l),$$

$$D(\lambda l) = 2(1 - \cos\lambda l\cosh\lambda l).$$

$$\lim_{\lambda\to 0} M_{\beta\alpha}(\lambda l) = \begin{bmatrix} 1 & 1/2 & 0 & -1/12 \\ 1/12 & 13/35 & -3/140 & -11/210 \\ 0 & -3/140 & 1/210 & 1/420 \\ -1/12 & -11/210 & 1/420 & 1/105 \end{bmatrix},$$

$$\lim_{\lambda\to 0} C_{\beta\alpha}(\lambda l) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 12 & -12 & -6 \\ 0 & -12 & 12 & 6 \\ 0 & -6 & 6 & 4 \end{bmatrix}.$$