A Theory for the Plastic Flow of Granular Materials

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A theory for the kinematics of granular materials is presented which satisfies the following properties: (1) It is expressed in the form of three-dimensional tensor equations; (2) the deformation is plastic, *i.e.*, there is no one-to-one correspondence between the stress and the strain-rate; (3) the material is isotropic, satisfying Saint Venant's principle; (4) the material is incompressible; and (5) the strain rate is derived from the associated flow rule. This theory thus resolves all the inconsistencies involved in the existing plasticity theories for granular materials. The basic idea is a new interpretation of the associated flow rule with geometric constraints of deformation. The elastic strain is then incorporated, and the theory is extended to an elastic-plastic theory.

I. INTRODUCTION

It has been shown that the statics of granular materials based on the Coulomb yield criterion provides a satisfactory basis for the analysis of limit equilibrium (*e.g.*, Sokolovskii[1]). However, various attempts to describe the velocity field have been less successful. Drucker and Prager[2] proposed the plasticity theory which had been developed for metals. They applied the associated flow rule to granular materials. Their theory then predicted unreasonable increase in specific volume during shear deformation, which has often been criticized in the past. Although various attempts have been made to modify or extend the Drucker-Prager theory (*e.g.*, Drucker,[3] Shield,[4-6] Drucker, Gibson and Henkel,[7] Jenike and Shield,[8] Jenike[9]), no remarkable consequences have been obtained. Spencer[10] introduced the idea that deformation occurs due to shear on certain critical planes, assuming that the strain-rate is dependent on the stress-rate as well as on the stress. This new approach has been further developed by Mandel,[11] Mandl and Fernández Luque,[12] de Josselin de Jong[13] Morisson and Richmond[14] and others,[15,16] and has been called the double-slip theory. However, their theories are almost limited to plane motions and are difficult to generalize in the form of three-dimensional tensor equations. Moreover, a great deal of sophistication is needed in the derivation of equations, and the results are very much complicated. Goodman and Cowin[17,18] assumed that the stress depends on the gradient of the solid volume fraction of the material as well as the strain-rate. However, their argument is restricted to formal continuum mechanical considerations, so that the constitutive equations are determined only by plausible assumptions. Kanatani[19-21] held a different viewpoint. He analyzed microscopic interparticle friction and collisions and took statistical average of the interactions to obtain constitutive equations for an equivalent continuum model of the flow.

In this paper, we reexamine the Drucker-Prager theory in the light of the above stated recent developments. We first consider the validity of the associated flow rule and show that the rule must be altered when applied to granular materials, while it remains valid when applied to metals. Then, we present a plasticity theory for granular materials which satisfies the following requirements: (1) All the equations are expressed in the form of three-dimensional tensor equations; (2) the deformation is plastic, *i.e.*, there is no one-to-one correspondence between the stress and the strain-rate, which implies that the components of the stress tensor are not mutually independent, and hence there must exist some relation among the stress components, which has usually been identified with the yield equation; (3) the material is isotropic, so that the stress tensor and the strain-rate tensor must have common principal axes (this requirement is referred to as Saint Venant's principle); (4) the material is incompressible; and (5) the strain-rate is derived from the associated flow rule. It has been believed that these requirements cannot be satisfied at the same time. In particular, requirements (4) and (5) have been believed to be unreconcilable, and requirment (3) has been doubted because it gives velocity characteristic surfaces different from the stress characteristic surfaces of limit equilibrium. Hence, which of them should be dropped has been the main concern in the arguments advanced so far (*e.g.*, Takagi,[22,23] Hythornthwaite,[24] Spencer,[15] Davis,[25] Mandl and Fernández Luque[12]). Our theory is the only one that meets all these requirements, though in a slightly different sense. Our theory for perfect plasticity turns out to be nothing but a straight extension of the Levi-Mises theory of metal plasticity. We then incorporate the elastic strain into the theory and develop an elastic-plastic theory for granular materials, which turns out to be an extension of the Prandtl-Reuss theory of metal plasticity.

II. ASSOCIATED FLOW RULE FOR GRANULAR MATERIALS

The associated flow rule is the rule that relates plastic strain increments and stresses through differentiation of the yield function. This process was first worked out in the theory of metal plasticity. Later, Drucker[26,27] formulated a fundamental postulate of material stability and derived the rule from his postulate. His postulate states that *when a body is in an arbitrary equilibrium, the work done by any cycle of application-and-removal of additional loading is non-negative.* Let the initial stress in an equilibrium at time $t = 0$ be σ^*_{ji}, and let t_1 designate the first occurrence of plastic strain. Furthermore, let the loading be continued until $t = t_2$, and let the removal of the added load take place until $t = t_3$ when the stress is again σ^*_{ji}. Since the work done by the stresses on the elastic strains during a closed cycle vanishes, and since plastic deformation occurs only during the interval $t_1 < t < t_2$, the work done by the additional loading is

$$\int_{t_1}^{t_2} (\sigma_{ji} - \sigma^*_{ji})\, \dot{e}^p_{ji}\, dt, \tag{1}$$

where e^p_{ji} is the plastic strain and the dot designates the time derivation. Throughout this paper, we adopt the Cartesian tensor notation and the rule of summation convention. Taking the limit $t_2 \to t_1$ and applying Drucker's postulate, we obtain inequality

$$(\sigma_{ji} - \sigma^*_{ji})\, \dot{e}^p_{ji} \geqq 0 \tag{2}$$

for an *arbitrary equilibrium stress* σ^*_{ji}. This is interpreted such that the angle made by the six-dimensional vectors $\sigma_{ji} - \sigma^*_{ji}$ and $\dot{e}^p_{ji}$ is not greater than $\pi/2$. Then, we can conclude that (i) the yield surface determined by the yield equation is convex; and (ii) if the surface is smooth, vector $\dot{e}^p_{ji}$ is normal to the yield surface at the point of σ_{ji}. Consequently, if the yield equation $f(\sigma_{ji}) = 0$ is regular, then we have

$$\dot{e}^p_{ji} = \Lambda \frac{\partial f}{\partial \sigma_{ji}}, \tag{3}$$

where Λ is a scalar quantity. This is also referred to as the normality condition.

Several alternatives have been proposed to Drucker's formulation. Il'iushin[28] asserted that the cycle of loading should be replaced by a cycle of the total strain. Yamamoto[29] and Green and Naghdi[30] gave formulations which include thermal effects. However, these alternatives result in a great deal of complexity and do not seem very successful. Meanwhile, the associated flow rule predicts unreasonable increase in specific volume when applied to granular materials whose yield function depends upon the hydrostatic pressure. The reason is easily understood if we consider the following example. Imagine a block on a horizontal plane as is shown in Fig. 1. If the friction between the block and the plane obeys the Coulomb law, the yield equation is $F = \pm\mu N$, where μ is the friction coefficient. As is seen from Fig. 2, the associated flow rule predicts normal displacements which do not actually occur. Suppose the block is in equilibrium under the normal force N^* and the horizontal force F^*. Let us apply a force F which has a backward horizontal component small in magnitude but has an upward normal component large enough in magnitude to cause slip (see Fig. 3). Then, remove the force to reduce the system in the initial state of equilibrium forces. The work done by the added force during this process is clearly negative because the block moves in the opposite direction to the horizontal component of F. (*The normal component of F does not do any work.*) Thus, Drucker's postulate is violated, and the system is not stable in the sense of Drucker. It is evident that if the block could move upward, Drucker's postulate would be satisfied. Thus, the increase in specific volume is inevitable if granular materials in which frictional slips occur are to be stable in the sense of Drucker. However, we assert here to draw an alternative conclusion: *Granular materials are no more stable in the sense of Drucker than the slip of a block on a plane.*

Frictional slip is a basic concept in the mechanics of granular materials, for the Coulomb yield criterion is derived from the local slip condition on potential slip-planes. If local slips are the mechanism of deformation, the specific volume must be conserved during deformations. This is a geometric constraint of deformation. Of course, the so called dilatancy may occur depending upon the configuration of constituent granules. But it cannot be described by the Coulomb yield criterion alone. In order to describe the dilatancy, we must introduce additional internal variables which express the internal state of the constituent granules such as the solid volume fraction introduced by Goodman and Cowin. If we are to regard the material as a continuous homogeneous body without such internal variables, we must necessarily assign the constraint of incompressibility.

Now, we try to modify Drucker's postulate so that it may be applied to plastic deformations with geometric constraints. A constraint of deformation defines an associated *constraining stress*, which is a portion of the stress that does not do any work for admissible deformations but does work only for virtual displacements violating the constraint. Since the constraints give an additional set of kinematic equations, the corresponding constraining stresses are treated as so many additional kinematic variables, which are often referred

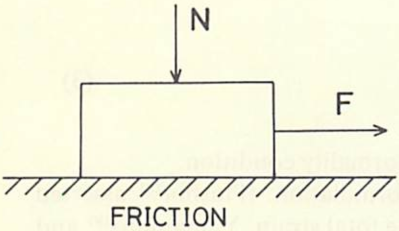

Fig. 1. Friction on a plane.

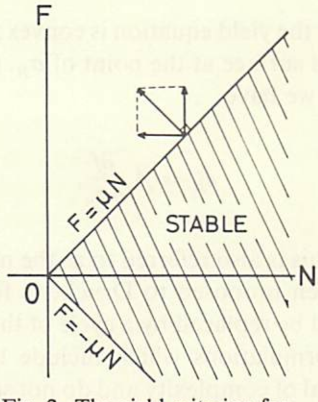

Fig. 2. The yield criterion for friction on a plane.

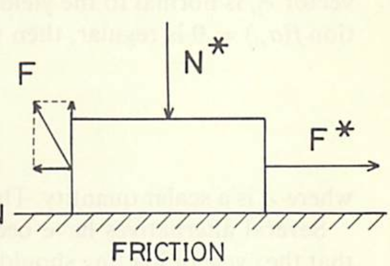

Fig. 3. A counter-example of Drucker's postulate.

to as *Lagrange multipliers* of the constraints. Hence, we must treat the constraining stresses separately from the remaining potential and dissipative stresses. Drucker's postulate is apparently a demand for the dissipative stresses. We now propose a modification to Drucker's postulate: *When a body is in an arbitrary equilibrium, the work done by any cycle of application-and-removal of loading such that the constraining stresses are kept constant is non-negative.* The normal force N in Fig. 1 is the constraining force for slips on the plane, and it is easily seen that this new postulate is satisfied by that system. The constraining stress corresponding to the constraint of incompressibility is simply the hydrostatic pressure $p=-(1/3)\,\sigma_{kk}$. Following the previous procedure, we again obtain inequality (2). However, the stress σ_{ji} on the yield surface is now linked with the initial stress σ^*_{ji} by a special stress-path along which p is kept fixed. Hence, *the choice of σ^*_{ji} is not arbitrary*, and consequences (i) and (ii) do not follow this time. Let us write the deviators of σ_{ji} and e_{ji}, respectively, as

$$\tilde{\sigma}_{ji}=\sigma_{ji}-\frac{1}{3}\,\delta_{ji}\,\sigma_{kk}, \quad \tilde{e}_{ji}=e_{ji}-\frac{1}{3}\,\delta_{ji}\,e_{kk}, \tag{4}$$

where δ_{ji} is the Kronecker delta. Inequality (2) is rewritten as

$$(\tilde{\sigma}_{ji}-\tilde{\sigma}^*_{ji})\,\dot{\tilde{e}}^p_{ji}+\frac{1}{3}\,(\sigma_{ll}-\sigma^*_{ll})\,\dot{e}^p_{kk}\geqq 0. \tag{5}$$

The second term vanishes according to our postulate. Hence, we have

$$(\tilde{\sigma}_{ji}-\tilde{\sigma}^*_{ji})\,\dot{\tilde{e}}^p_{ji}\geqq 0 \tag{6}$$

for an arbitrary stress deviator $\tilde{\sigma}^*_{ji}$ that gives equilibrium stress for a given fixed p. Write the yield equation in the form $f(\tilde{\sigma}_{ji},p)=0$. Then we can conclude the *normality condition*

$$\dot{\tilde{e}}^p_{ji}=\Lambda\,\frac{\partial f}{\partial\sigma_{ji}}\bigg|_p \tag{7}$$

where $\partial/\partial\sigma_{ji}|_p$ designates the *partial differentiation with fixed p.* The fact that the right-hand side gives the deviator components is readily seen from

$$\left.\frac{\partial f}{\partial \sigma_{ji}}\right|_p = \frac{\partial f}{\partial \bar{\sigma}_{lk}} \frac{\partial \bar{\sigma}_{lk}}{\partial \sigma_{ji}} = \frac{\partial f}{\partial \bar{\sigma}_{lk}} (\delta_{lj}\, \delta_{ki} - \delta_{ji}\, \delta_{lk}). \tag{8}$$

We have now reached a new expression of the associated flow rule for materials for which only incompressible plastic deformations are admitted. Note that *application of this new rule to metal plasticity does not bring about any modifications to the existing theories*, because it has been assumed that the yield function for metals does not depend on the hydrostatic pressure.

III. EQUATIONS FOR PERFECT PLASTIC FLOWS OF GRANULAR MATERIALS AND CHARACTERISTIC SURFACES

In the following, we denote the strain-rate tensor by E_{ji}, *i.e.*,

$$E_{ji} = \partial_{(j} v_{i)} (= \dot{e}_{ji}), \tag{9}$$

where v_i is the velocity vector and ∂_i designates $\partial/\partial x_i$. Here, ( ) indicates the symmetrization of indices. We first investigate the perfect or rigid plastic flow, so that E_{ji} is the plastic strain-rate. As the yield equation, we adopt the extended von Mises equation

$$f(\bar{\sigma}_{ji}, p) \equiv \sqrt{(1/2)\, \bar{\sigma}_{ji}\, \bar{\sigma}_{ji}} - \alpha p - k = 0, \tag{10}$$

which Drucker and Prager[2)] introduced. Equation (10) is also a natural outcome of Kanatani's statistical theory of particle flow.[21)] Application of the associated flow rule in our sense to Eq. (10) yields

$$E_{ji} = \frac{\Lambda}{2} \frac{\bar{\sigma}_{ji}}{\alpha p + k}. \tag{11}$$

The scalar quantity Λ is determined by solving Eq. (11) in terms of $\bar{\sigma}_{ji}$ and substituting it in Eq. (10). We obtain $\Lambda = \sqrt{2 E_{ji} E_{ji}}$. Thus, we have the following constitutive equation.

$$\bar{\sigma}_{ji} = \frac{\alpha p + k}{\sqrt{(1/2) E_{lk} E_{lk}}} E_{ji}. \tag{12}$$

The right-hand side is a homogeneous form of degree 0 in E_{ji}, and hence there exists no one-to-one correspondence between the stress and the strain-rate. Clearly, Saint Venant's principle and material incompressibility are satisfied.

Consider the plane motions in order to express the two constants α and k in terms of the internal friction angle ϕ and the cohesion constant c. We obtain, from the diagram of the Mohr stress-cycle, $\alpha = \sin\phi$ and $k = c \cos\phi$. These expressions are different from those of Drucker and Prager[2)] because of our assumption of incompressibility. If $\alpha = 0$ in Eq. (12), the effects of hydrostatic pressure vanish and Eq. (12) reduces to the Levi-Mises equation of metal plasticity. If, on the other hand, the cohesion k is neglected, Eq. (12) reduces to the equation of Kanatani[19,21)] derived for flows of cohesionless rigid particles by a statistical method. We call Eq. (12) the extended Levi-Mises equation.

The equation of motion for a continuum in general has the form

$$\rho \frac{\mathrm{d}v_i}{\mathrm{d}t} = \partial_j \sigma_{ji} + \rho b_i, \tag{13}$$

where ρ is the density and b_i is the body force per unit mass. Here $\mathrm{d}/\mathrm{d}t$ represents the Lagrange derivative $\partial/\partial t + v_i \partial_i$. Equation (13), into which Eq.(12) is substituted, and the equation of incompressibility give a set of four equations for four independent variables v_x, v_y, v_z and p as follows:

$$\rho \frac{\mathrm{d}v_i}{\mathrm{d}t} = -\partial_i p + \frac{\alpha}{\Lambda} E_{ji} \partial_j p + \frac{\alpha}{2\Lambda} \bar{p} \Delta v_i - \frac{\alpha}{2\Lambda^3} \bar{p} E_{ji} E_{lk} \partial_j \partial_l v_k + \rho b_i, \qquad \partial_i v_i = 0, \tag{14}$$

where $\bar{p} = p + k/\alpha$, $\Lambda = \sqrt{(1/2) E_{ji} E_{ji}}$ and Δ is the Laplacian operator.

Characteristic surfaces for a set of partial differential equations are defined as follows (*cf.* Thomas[31]): Suppose all the values of partial derivatives appearing in the equations except those of the highest rank of differentiation for each quantity are specified on a certain surface. In general, the set of equations determines the remaining values of the highest derivatives. If they are indeterminate in particular, the surface is said to be a characteristic surface for the given data. Then, one cannot integrate the equations by specifying boundary conditions on that surface. This means that discontinuity in the highest derivatives can arise across the surface even if all the remaining quantities are continuous. Hence, we can obtain the characteristic surface by applying the geometrical conditions of discontinuity (*cf.* Thomas[31]). In the case of plane motions, we get

$$\theta = \pm \pi/4, \pm (\pi/4 - \phi/2), \tag{15}$$

where θ is the angle between the surface normal and the principal axis of minimum compression. (For details, see Kanatani.[32]) We have thus obtained two types of characteristic surfaces. One is the surfaces of maximum shearing stress which make the angle $\pi/4$ to the principal stress axes. The other is the characteristic surfaces of limit equilibrium. Now we have shown that the material isotropy, the material incompressibility and the associated flow rule, all of which the extended Levi-Mises equation exibits, are compatible with the experimental fact that discontinuity is observed across the surfaces of maximum shearing during plane motion. We can also show that the pressure discontinuity vanishes on the surface of maximum shearing for plane motion. Hence, only discontinuity with regard to the velocity field is possible across those surfaces, which again agrees with the consequences of the statics of limit equilibrium.

IV. ELASTIC-PLASTIC THEORY OF GRANULAR MATERIALS AND PROPAGATION OF SINGULAR SURFACES

We now extend the previous results to an elastic-plastic theory, incorporating the elastic strain as well. Let the total strain-rate E_{ji} be decomposed into the elastic part E^e_{ji} and the plastic part E^p_{ji}, *i.e.*,

$$E_{ji} = E^e_{ji} + E^p_{ji} \tag{16}$$

in such a way that the elastic strain-rate E^e_{ji} determines the stress-rate $D\sigma_{ji}/Dt$. For simplicity, we assume linearity and put

$$\frac{D\sigma_{ji}}{Dt} = 2\mu E^e_{ji} + \lambda\delta_{ji}E^e_{kk}, \tag{17}$$

where μ and λ are constants. In order that this expression be invariant to translations and rigid rotations of the coordinate system, the time derivative $D\sigma_{ji}/Dt$ must be interpreted as

$$\frac{\partial\sigma_{ji}}{\partial t} + v_k\partial_k\sigma_{ji} - \sigma_{ki}\partial_{[k}v_{j]} - \sigma_{jk}\partial_{[k}v_{i]}, \tag{18}$$

where [ ] designates the alternation of indices. This time derivative is called the covariant time derivative (Thomas[31]) or the Jaumann-Noll derivative (Eringen[33]). Taking the deviator and the trace of Eq. (17), we can rewrite it as

$$\frac{D\tilde{\sigma}_{ji}}{Dt} = 2\mu\,\tilde{E}^e_{ji}, \quad \frac{Dp}{Dt} = -\kappa E^e_{kk}, \tag{19}$$

where $\kappa(= (2\mu + 3\lambda)/3)$ is the bulk modulus of the material. The plastic strain-rate is, on the other hand, given by the associated flow rule in our sense, *i.e.*,

$$\tilde{E}^p_{ji} = \Lambda\frac{\partial f}{\partial\sigma_{ji}}\bigg|_p, \quad E^p_{kk} = 0. \tag{20}$$

Combination of Eqs. (19) and Eqs. (20) yields

$$\frac{D\sigma_{ji}}{Dt} = 2\mu\left(\tilde{E}_{ji} - \Lambda\frac{\partial f}{\partial\sigma_{ji}}\bigg|_p\right), \quad \frac{Dp}{Dt} = -\kappa\,E_{kk}. \tag{21}$$

This set of equations is an extension of the Prandtl-Reuss equation of metal plasticity, and hence we call them extended Prandtl-Reuss equations. The scalar Λ is determined by taking the derivative Df/Dt of the yield function. Then we get[32]

$$\Lambda = \frac{E_{ji}\partial f/\partial\sigma_{ji}|_p - (\kappa/2\mu)\,E_{kk}\partial f/\partial p}{\partial f/\partial\sigma_{lk}|_p\partial f/\partial\sigma_{lk}|_p}. \tag{22}$$

The equation of continuity and the equations of motion are

$$\frac{d\rho}{dt} + \rho\partial_i v_i = 0, \quad \rho\frac{dv_i}{dt} = \partial_j\sigma_{ji} + \rho b_i, \tag{23}$$

respectively. Equations (21) and Eqs. (23) provide ten equations for ten unknown variables ρ, v_i, p and $\tilde{\sigma}_{ji}$.

From this set of equations, we can investigate the propagation of singular surfaces. We assume that the derivatives appearing in Eqs. (21) and (23) are discontinuous across a surface which is moving in the direction of its unit normal. Applying the kinematic compatibility conditions of discontinuity (*cf.* Thomas,[31] Eringen[33]), we can determine the velocity of propagation, adopting the extended von Mises equation (10) as the yield equation. (For details, see Kanatani.[32]) If, in particular, the singular surface is stationary, we have in the case of plane motions

$$\cos 2\theta = 0,\ \alpha,\ \alpha\kappa/(\kappa + \mu/3), \tag{24}$$

where θ is the angle between the surface normal and the principal axis of minimum compression in the deformation plane. The former two coincide with the characteristic surfaces of perfect plastic flows. The angle θ determined by the last one approaches $\pm (\pi/4 - \phi/2)$ in the limit of incompressibility, *i.e.*, $\kappa/\mu \to \infty$ (or equivalently, $\nu \to 0.5$, where $\nu = \lambda/2(\lambda + \mu) = (3\kappa - 2\mu)/2(3\kappa + \mu)$ is the Poisson ratio). Especially, *if the singular surface separates a region of plastic flow from a region in elastic limit equilibrium, the surface must be one of these three types.*

V. CONCLUDING REMARKS

We have presented a plasticity theory for the kinematics of granular materials based on a new interpretation of the associated flow rule with geometric constraints. Adopting the extended von Mises equation as the yield equation, we have obtained the extended Levi-Mises equation in the form of a three-dimensional tensor equation, which exibits perfect plasticity, incompressibility, isotropy and Saint Venant's principle. We have shown that the characteristic surfaces obtained in the statics of granular materials are also the characteristic surfaces of the velocity field. At the same time, we have shown that the surfaces of maximum shearing stress in the case of plane motion are also the characteristic surfaces in accordance with experimental observations. Then, we have extended the theory to an elastic-plastic theory, incorporating the elastic strain as well, and have obtained the extended Prandtl-Reuss equations. We have also discussed briefly the types of singular surfaces. Thus, our theory is a straight extension of the statics of limit equilibrium to kinematics.

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