A Continuum Theory for the Flow of Granular Materials

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The flow of granular materials is modeled by a polar continuum, the rotation of particles being described as an additional tensor field. Kinematic equations are derived by means of the couple-stress theory. In order to obtain constitutive equations, the microscopic particle friction is analyzed, and the macroscopic relations are deduced by statistical averaging. Macroscopically equivalent stresses are determined for incompressible granular flows by the use of the energy dissipation relation. Inclined gravity flows are studied to make clear the essential characteristics of the granular flow. The existence of the similarity relation and the angle of repose are the most remarkable consequences of our theory. Some numerical analyses are also given.

I. INTRODUCTION

A granular material is an aggregate of a large number of particles, and its mechanical properties have been studied in soil mechanics and powder mechanics. In soil mechanics, static stress analysis, especially that of the limiting equilibrium, has been closely investigated by both experimental and theoretical approaches.[1–3] Powder mechanics deals chiefly with conveyance of granular materials by various equipments.[4,5] However, general dynamic theories of granular flows have not been fully developed due to the internal complexity of the materials. Several authors described granular flows by models with some specific characteristics taken into consideration. Goodman and Cowin[6,7] took the voidage of the material as an additional field variables and developed a general continuum theory. However, though their theory is formally consistent, various material constants appearing in the theory are not deducible on the basis of the microscopic properties of the constituent granules. In this paper, we take the statistical approach used in statistical mechanics: we first study the microscopic interactions among particles and then deduce macroscopic relations by statistical averaging. We assume that particles are rigid spheres of radius a and mass m, and that pneumatic or hydrodynamic effects are absent. The only possible interactions among particles are assumed to be those of friction. Furthermore, the flow is assumed to be fairly ordered, so that it can be approximately regarded as an incompressible flow. We regard the rotation of particles as a tensor field quantity and derive basic kinematic equations by the couple-stress theory of polar continua. Then, we analyze the microscopic friction among particles and deduce macroscopic relations by means of the statistical method. Equivalent stresses are determined by the macroscopic energy dissipation relation of the flow. Finally, the inclined gravity flow is studied to make clear the essential characteristics of the granular flow. The existence of the similarity relation and the angle of repose are the most remarkable consequences of our theory. Some numerical analyses are also given.

II. KINEMATIC EQUATIONS OF POLAR CONTINUUM

We choose the velocity v^i and the rotation velocity ω_{ji}, which is an antisymmetric tensor, of particles as basic kinematic field variables, so that the continuum is regarded as a polar continuum. The theory of polar continua, or the couple-stress theory, was first developed as an extension of the classical theory of elasticity.[8,9] Oshima,[10] Eringen[11,12] and others[13] applied the theory to flows of fluids. Let us first write down basic kinematic equations, following these theories. The generalized forces corresponding to the kinematic variables v^i and ω_{ji}, are the usual force and the moment or the couple, respectively. These forces are exerted as body forces and surface forces. We assume that the usual body force ρb^i is that of gravity alone and that the body couple is absent. As the surface forces, we must consider both the usual stress σ^{ji} and the couple-stress μ^{kji}. The conservation laws for mass, momentum and angular momentum are written as

$$\frac{d}{dt}\int \rho dV = 0, \qquad \frac{d}{dt}\int \rho v^i dV = \int \rho b^i dV + \int \sigma^{ji} n_j dS, \tag{1),(2}$$

$$\frac{d}{dt}\int \left[2\rho x^{[j} v^{i]} + \frac{2}{5}\rho a^2 \omega^{ji}\right] dV = \int 2\rho x^{[j} b^{i]} dV + \int [2x^{[j}\sigma^{|k|i]} n_k + \mu^{kji} n_k]\, dS, \tag{3}$$

where the region of integration is any part of the material moving in the flow and n_i is the unit normal to the surface of the region. Throughout this paper, we adopt the rule of summation convention and denote $\partial/\partial x^i$ by ∂_i. The coordinate system is always Cartesian, so that we do not make any distinction between contravariant and covariant components of tensors. As usual, we use ( ) and [ ] to indicate the symmetric and the antisymmetric components of tensors respectively, *i.e.*,

$$A_{(ji)} = \frac{1}{2}(A_{ji} + A_{ij}), \qquad A_{[ji]} = \frac{1}{2}(A_{ji} - A_{ij}).$$

Applying Gauss' theorem, we obtain the following differential equations:

$$\frac{d\rho}{dt} + \rho\partial_k v^k = 0, \qquad \rho\frac{dv^i}{dt} = \partial_j \sigma^{ji} + \rho b^i, \tag{4),(5}$$

$$\frac{2}{5}\rho a^2 \frac{d\omega^{ji}}{dt} = \partial_k \mu^{kji} + 2\sigma^{[ji]}, \tag{6}$$

where $d/dt = \partial/\partial t + v^k\partial_k$.

Now, consider the conservation law for energy. Since there is no potential force acting between particles and the particles are rigid, we need not consider any potential energy. Hence, if $d'W/dt$ is the rate of work done by external forces and K is the kinetic energy of the region, we have

$$\frac{d'W}{dt} - \frac{dK}{dt} = \int \Phi dV, \tag{7}$$

where Φ is the rate of energy dissipation in a unit volume. We call Φ the dissipation function, though 2Φ is customarily so named. Substituting

$$\frac{d'W}{dt} = \int \rho b^i v_i dV + \int \left[\sigma^{ji} n_j v_i + \frac{1}{2} \mu^{kji} n_k \omega_{ji} \right] dS, \tag{8}$$

$$\frac{dK}{dt} = \frac{d}{dt} \int \left[\frac{1}{2} \rho v^i v_i + \frac{1}{10} \rho a^2 \omega_{ji} \omega^{ji} \right] dV, \tag{9}$$

into (7), and using Gauss' theorem again, we finally obtain

$$\Phi = \tilde{\sigma}^{ji} \partial_j v_i - \sigma^{[ji]} \omega_{ji} + \frac{1}{2} \mu^{kji} \partial_k \omega_{ji},$$

$$\tilde{\sigma}^{ji} \equiv \sigma^{(ji)} - \frac{1}{3} \delta_{ji} \sigma^{kk}. \tag{10}$$

In the usual theory of polar fluids, Φ is formally assumed to be a quadratic form in $\partial_j v_i$, ω_{ji} and $\partial_k \omega_{ji}$. Then, linear constitutive equations are obtained. However, we take, in this paper, the statistical approach; we first consider the microscopic energy dissipation due to friction of particles and deduce Φ by the statistical method. Suppose we get

$$\Phi = \Phi(X_1, X_2, \ldots, X_N), \tag{11}$$

where the arguments are generalized deformation rates. In general, if some particular mechanism of energy dissipation is assumed, the form of Φ becomes homogeneous in its arguments. The degree of homogeneity is characteristic to the corresponding mechanism of dissipation. Let k be the degree of homogeneity. According to Euler's theorem, we have

$$\Phi = \frac{1}{k} \sum_{\alpha=1}^{N} \frac{\partial \Phi}{\partial X_\alpha} X_\alpha. \tag{12}$$

We can then define $(1/k)\partial\Phi/\partial X_\alpha$ as the generalized stress corresponding to the generalized deformation rate X_α. The stress thus defined is considered to well characterize the internal dissipation mechanism. We call this procedure the Euler decomposition of the dissipation function and $k - 1$ the degree of dissipation.

III. MICROSCOPIC MODEL OF PARTICLE FRICTION

Consider two rotating particles in contact with each other as shown in Fig. 1. We can put, in the statistical average.

$$v_i' = v_i + 2an_j D_{ji}, \qquad \omega_{ji}' = \omega_{ji} + 2an_k \Omega_{kji}, \tag{13}$$

where

$$D_{ji} = \partial_j v_i, \qquad \Omega_{kji} = \partial_k \omega_{ji}, \tag{14}$$

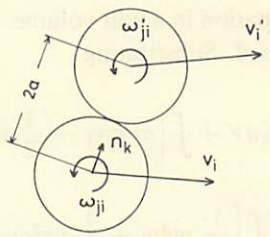

Fig. 1. Friction between two rotating spheres.

are the macroscopic gradients of the velocity and the rotation velocity, respectively. The tangential component of the relative velocity at the friction point is

$$\xi_i = 2a(n_j D_{ji} - n_j \omega_{ji} - n_k n_j n_i D_{kj} - a n_k n_j \Omega_{kji}). \tag{15}$$

Now, the direction n_i to the contact point is completely random, so that it is regarded as a random variable whose probability distribution is uniform over the whole solid angle. Let us calculate the root mean square of the magnitude of ξ_i. Making use of formulae

$$\overline{n_j n_i} = \frac{1}{3}\delta_{ji}, \qquad \overline{n_k n_j n_i} = 0, \qquad \overline{n_m n_l n_k n_j n_i} = 0,$$

$$\overline{n_l n_k n_j n_i} = \frac{1}{15}(\delta_{lk}\delta_{ji} + \delta_{lj}\delta_{ki} + \delta_{li}\delta_{kj}), \tag{16}$$

where the bar means the average over the whole solid angle, and δ_{ji} is the Kornecker delta, we obtain

$$\sqrt{\overline{\xi_i \xi_i}} = \frac{2\sqrt{6}}{3} a\hat{\omega}, \tag{17}$$

where we have put

$$\hat{\omega} = \sqrt{\frac{3}{10}E_{ji}E_{ji} + \frac{1}{2}R_{ji}R_{ji} + \frac{a^2}{10}(\Omega_{kkj}\Omega_{lij} + \Omega_{kji}\Omega_{kji} + \Omega_{kji}\Omega_{jki})},$$

$$E_{ji} \equiv D_{(ji)} - \frac{1}{3}\delta_{ji}D_{kk}, \qquad R_{ji} \equiv \omega_{ji} - D_{[ji]}. \tag{18}$$

Let f_κ ($\kappa = 1,2,3, \ldots, n$) be the normal component of the contact force at the κ-th contact point and put $\bar{f} = \Sigma f_\kappa / n$. In order to determine $\bar{f}$, consider virtual compression by the ratio α (Fig. 2). The virtual work done by f_κ's is $(1/3)an\bar{f}\alpha$ per a single particle, which means the virtual work done in a unit volume is $(1/3)n\bar{f}(\rho/m)a\alpha$. This work must be equal to the work $p\alpha$ done by the macroscopic pressure p because of the definition of p as the Lagrange multiplier with respect to the constraint of incompressibility. Hence, we have

$$\bar{f} = \frac{3mp}{n\rho a} \tag{19}$$

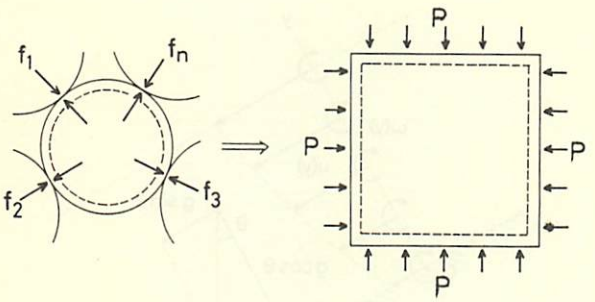

Fig. 2. The contact forces and the macroscopic pressure.

The average rate of energy dissipation at the κ-th contact point can be approximated by $\mu \bar{f}_\kappa \xi$, where μ is the kinetic friction coefficient. Then, the rate of energy dissipation for a single particle is $n\mu \bar{f} \xi$. Multiplying ρ/m, the number of particles in a unit volume, and dividing it by 2, because each contact is doubly counted, we finally obtain the energy dissipation rate Φ per unti volume in the form

$$\Phi = \sqrt{6}\, \mu p \hat{\omega}. \tag{20}$$

Following the procedure discussed in the previous section, we have constitutive equations as follows:

$$\begin{aligned} \tilde{\sigma}^{ji} &= \frac{3\sqrt{6}\,\mu}{10} \frac{p}{\hat{\omega}} \left(\partial_{(j} v_{i)} - \frac{1}{3} \delta_{ji} \partial_k v^k \right), \\ \sigma^{[ji]} &= \frac{\sqrt{6}\,\mu}{2} \frac{p}{\hat{\omega}} (\partial_{[j} v_{i]} - \omega_{ji}), \\ \mu^{kji} &= \frac{\sqrt{6}\,\mu a^2}{5} \frac{p}{\hat{\omega}} (\delta_{k[j} \partial_{l} \omega_{l|i]} + \partial_k \omega_{ji} - \partial_{[j} \omega_{i]k}). \end{aligned} \tag{21}$$

We can easily see that the degree of dissipation is zero and that the stresses have homogeneous forms of degree 0 in $\partial_j v_i$, ω_{ji}, and $\partial_k \omega_{ji}$.

IV. INCLINED GRAVITY FLOW

Consider an infinite slab of granular material of thickness h inclined at an angle θ to the gravity field and having a free upper surface while supported below by a flat plate as is shown in Fig. 3. Let us take the natural units with respect to the density ρ, the thickness h and the acceleration of gravity g, *i.e.*, we measure all length in terms of h, all masses in terms of ρh^3 and the time in terms of $\sqrt{h/g}$. The equations of motion are reduced to

$$\frac{du}{dt} = \sin\theta + \frac{d\sigma^{yx}}{dy}, \quad 0 = -\cos\theta - \frac{dp}{dy}, \quad \frac{d\omega}{dy} = \frac{5}{2a^2} \left(2\sigma^{[xy]} + \frac{d\mu^{yxy}}{dy} \right). \tag{22}$$

The second one is integrated to yield $p = (1 - y)\cos\theta$, and hence the constitutive equations become as follows:

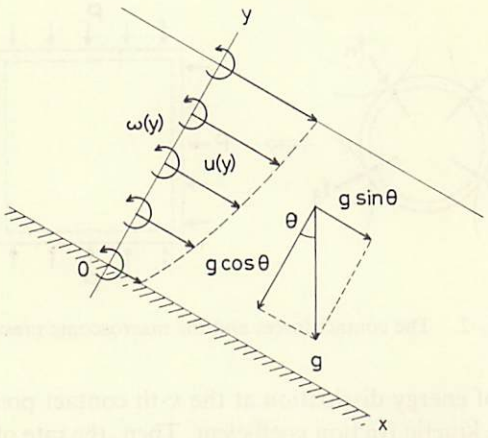

Fig. 3. The inclined gravity flow of granular materials.

$$\hat{\omega} = \sqrt{\left(\frac{1}{2}\frac{\partial u}{\partial y} + \omega\right)^2 + \frac{3}{20}\left(\frac{\partial u}{\partial y}\right)^2 + \frac{2}{5}a^2\left(\frac{\partial \omega}{\partial y}\right)^2}, \tag{23}$$

$$\begin{aligned}
\sigma^{yx} &= \sqrt{6}\,\mu\cos\theta\,\frac{1-y}{\hat{\omega}}\left(\frac{2}{5}\frac{\partial u}{\partial y} + \frac{1}{2}\omega\right),\\
\sigma^{[yx]} &= -\sqrt{6}\,\mu\cos\theta\,\frac{1-y}{\hat{\omega}}\left(\frac{1}{4}\frac{\partial u}{\partial y} + \frac{1}{2}\omega\right),\\
\mu^{yxy} &= \frac{\sqrt{6}}{10}\mu\alpha^2\cos\theta\,\frac{1-y}{\hat{\omega}}\frac{\partial \omega}{\partial y}.
\end{aligned} \tag{24}$$

If the flow is seady, Eqs. (22) are reduced, on substitution of (24), to

$$\begin{aligned}
\frac{d}{dy}\left(\frac{1-y}{\hat{\omega}}\left(\frac{2}{5}\frac{du}{dy} + \frac{1}{2}\omega\right)\right) &= -\frac{\tan\theta}{\sqrt{6}\,\mu}\\
\frac{d}{dy}\left(\frac{1-y}{\hat{\omega}}\frac{d\omega}{dy}\right) &= \frac{20}{a^2}\frac{1-y}{\hat{\omega}}\left(\frac{1}{4}\frac{du}{dy} + \frac{1}{2}\omega\right).
\end{aligned} \tag{25}$$

From these equations, we can conclude:

Theorem of Similarity

If u and ω satisfy the equations of steady flow, so do Au and $A\omega$, where A is an arbitrary constant.

This theorem is one of the consequences of the fact that the internal energy dissipation is caused by friction, whose magnitude does not depend upon the friction velocity.

Now, consider the simple shear flow with passive rotations of particles:

$$u = Ay, \qquad \omega = -\frac{1}{2}A, \qquad (A = \text{const.}) \tag{26}$$

Theorem of the Angle of Repose

Steady flow is possible only when θ, the angle of inclination, is equal to θ^*, where

$$\theta^* = \tan^{-1}\left(\frac{3\sqrt{10}}{10}\mu\right). \tag{27}$$

If $\theta > \theta^*$, the flow is accleerated, and if $\theta < \theta^*$, the flow is damped.

Hence, the angle θ^* is regarded as the kinetic angle of repose. If $\mu = 0.7$ for glass, then $\theta^* = 33.6°$, which is a quite reasonable value[4].

Suppose that the stresses at the lower boundary are given by (24), where $\partial u/\partial y$, ω and $\partial\omega/\partial y$ are replaced by $u(0)/2a$, $\omega(0)$ and $\omega(0)/2a$ respectively. Let $\theta = \theta^* = 30°$. Since equations (25) are non-linear and difficult to solve analytically, we execute computer simulation of equation (22), taking the initial flow to be (26) with $A = 1$. The profiles of the velocity and the rotation velocity are shown in Fig.4 and Fig.5, respectively. We can see that the interaction at the boundary affects the flow near the boundary, and that the thickness of the 'boundary layer' increases as a/h increases.

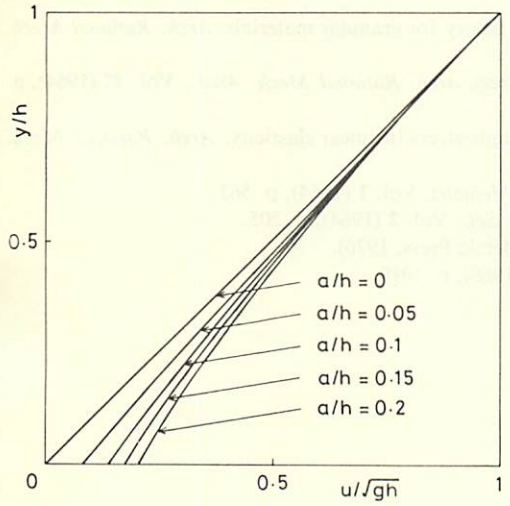

Fig. 4. The velocity profile of the inclined gravity flow.

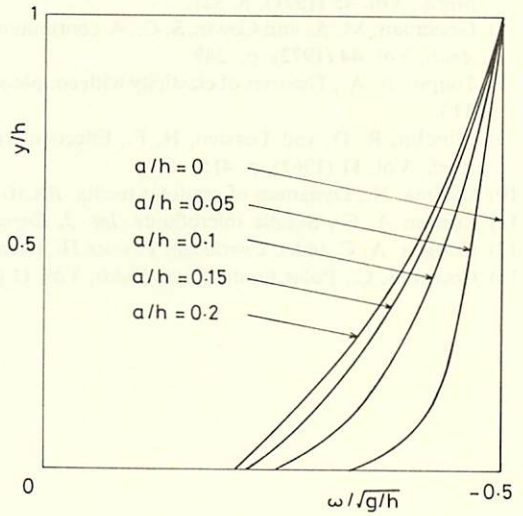

Fig. 5. The rotation velocity profile of the inclined gravity flow.

V. CONCLUDING REMARKS

We have established a polar continuum model for the flow of granular materials. Basic kinematic equations are derived on the basis of the couple-stress theory. In order to obtain constitutive equations, we examined a microscopic mechanism of energy dissipation due to friction of particles. Then, we deduced macroscopic dissipation relations by the statistical method. Macroscopic stresses are determined by the Euler decomposition of the dissipation function. We studied the inclined gravity flow and showed that the existence of the similarity

theorem and the angle of repose are natural consequences of our theory. Some numerical analyses are given with regard to the effects of boundary conditions.

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