A Continuum Theory for the Flow of Granular Materials (II)

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A continuum theory for slow flows of granular materials was developed in Part 1. The theory is extended in this part to include fast flows in which particle collisions play an important role. The particle fluctuations are regarded as macroscopic "heat", and a thermodynamic analogy is developed. The "equations of state" are determined by assuming "local equilibrium", and the normal stress effects due to the "thermal dilatation" are shown to exist. Finally, "entropy" is introduced, and the law of "entropy" increase is formulated.

I. INTRODUCTION

In Part 1[1], we have established a polar continuum model for the flow of granular materials. The materials were assumed to consist of cohesionless rigid spheres of uniform size and weight, and the flow was assumed to be fairly ordered, so that possible interactions among particles were assumed to be those of friction alone. We term such flows *slow flows.* In this part, we study those flows in which particle collisions play an important role. We call such flows *fast flows.* The existence of these two distinct flow regimes has been revealed by experiments of gravity flows in an inclined chute[2-5]. When the inclination angle is small, the particles are rolling in an ordered manner. When the angle reaches a critical value, the particles begin chaotic fluctuations, interacting vigorously with their neighbors. We regard the particle fluctuations as macroscopic "heat", and develop a thermodynamic analogy. We assume "local equilibrium" to deduce "equations of state". "Entropy" is also defined, and the law of "entropy" increase is formulated. We analyze the "thermal dilatation" of the flow due to the particle collisions, which demonstrates the *normal stress effects* of granular materials observed by Bagnold[6,7] and Savage[8]. As we did in Part 1, we adopt the index notation of tensors and the rule of summation convention throughout this paper. The coordinate system is always Cartesian, so that we do not make any distinction between contravariant and covariant components of tensors.

II. PARTICLE FLUCTUATIONS AND A THERMODYNAMIC ANALOGY

Let $v_i(P)$ and $\omega_{ji}(P)$ be the velocity and the angular velocity, respectively, of a particle at the point P. We put

$$v_i(P) = v_i + v_i'(P), \quad \omega_{ji}(P) = \omega_{ji} + \omega_{ji}'(P), \tag{1}$$

where v_i and ω_{ji} are the averaged values over some neighborhood around the point P, while $v_i'(P)$ and $\omega_{ji}'(P)$ are random variables of zero mean representing the irregular fluctuations due to the interparticle collisions. The kinetic energy averaged over the neighborhood is

$$\frac{1}{2}\rho v_i v_i + \frac{1}{10}\rho a^2 \omega_{ji}\omega_{ji} + \frac{1}{2}\rho \overline{v_i' v_i'} + \frac{1}{10}\rho a^2 \overline{\omega_{ji}' \omega_{ji}'}, \tag{2}$$

where ρ is the bulk density of the particle assembly, a is the radius of the particle, and the bar means the average over the neighborhood. The first two terms of expression (2) are the kinetic energy density of the material in the continuum description. The remaining terms can be regarded as "internal energy" of the material. Being an averaged quantity, it is a continuous quantity. We define the "internal energy" per unit mass to be

$$\varepsilon = \frac{1}{2}\overline{v_i' v_i'} + \frac{1}{10}a^2 \overline{\omega_{ji}' \omega_{ji}'}. \tag{3}$$

The existence of the fluctuations does not affect the conservation laws (1)–(3) in Part 1, for they are expressions for the mean values. Hence, we have Eqs. (4)–(6) in Part 1 also for fast flows.

Consider the energy conservation law. We look on the irregular fluctuations of the particles as "heat". We have

$$\frac{\mathrm{d}K}{\mathrm{d}t} + \frac{\mathrm{d}U}{\mathrm{d}t} = \frac{\mathrm{d}'W}{\mathrm{d}t} + \frac{\mathrm{d}'Q}{\mathrm{d}t}, \tag{4}$$

where $\mathrm{d}'W/\mathrm{d}t$ and $\mathrm{d}K/\mathrm{d}t$ are given by Eqs. (8) and (9) in Part 1, respectively, and

$$U = \int \rho\varepsilon \mathrm{d}V, \quad \frac{\mathrm{d}'Q}{\mathrm{d}t} = \int q \mathrm{d}V - \int h^i n_i \mathrm{d}S. \tag{5), (6}$$

Here q is the "heat" supply per unit volume, *i.e.*, $-q$ is the rate of energy dissipation in a unit volume, while h^i is the "heat flux", *i.e.*, the transmission of the fluctuation energy to neighboring particles. Substitution of these equations in (4) with application of integration by parts yields

$$\rho\frac{\mathrm{d}\varepsilon}{\mathrm{d}t} = -p\partial_i v^i + \Phi + q - \partial_i h^i, \tag{7}$$

where $p = -(1/3)\sigma^{kk}$, and Φ is the form defined by Eq. (10) in Part 1. Now we can expect that the particle fluctuations soon grow up to an extent determined by the macroscopic flow pattern. We say that the flow is in "local equilibrium", if the "internal energy" ε depends not on the history of motion but only on the present state of the flow in the form $\varepsilon = \varepsilon(v_i, \partial_j v_i, \omega_{ji}, \partial_k \omega_{ji})$.

In a "local equilibrium" flow, the work done by the stresses are classified into two categories. One is the "thermal work" done by the pressure p caused by the interparticle collisions. This work directly changes the "internal energy" of the material. The other is the "dissipative work" spent in the interparticle friction and collisions. We assume that, in "local equilibrium", the fluctuations of a particle are nearly the same as those of surrounding ones, so that the "heat flux" vanishes in "local equilibrium". Equation (7) is then decomposed into two parts such as

$$\rho \frac{d\varepsilon}{dt} = -p\partial_i v^i, \quad -q = \Phi. \tag{8}$$

III. EQUATIONS OF STATE AND CONSTITUTIVE EQUATIONS

We consider those flows whose bulk density ρ is close to ρ_0, the bulk density of randomly packed spheres. This implies that each particle in the flow is almost always repeating collisions to the nearest particles. Consider in the flow a region containing several particles. If m is the mass of one particle, the volume of the region divided by the number of particles in it is $V = m/\rho$, which is the volume assigned to one particle. Let us call it the *occupation volume* of the particle. Define the *occupation radius* $r = (3V/4\pi)^{1/3}$ as the radius of a sphere whose volume equals the occupation volume. Now, a particle is assumed to be repeating collisions against a rigid spherical wall of radius r, the occupation radius, with fluctuation velocity v' (see Fig. 1). Let r_0 be the occupation radius of randomly packed spheres. The particle travels without collisions over the distance $2(r - r_0)$. Hence, it collides against the wall $v'/2(r - r_0)$ times per unit time. The momentum given to the wall is $2mv'$ for each collision. Hence, the pressure on the wall is $p = mv'^2/4\pi r^2(r - r_0) = \rho\rho_0 v'^2/3(\rho_0 - \rho^{1/3}\rho_0^{2/3})$. Since $\rho \cong \rho_0$, we can expand the denominator; $\rho_0 - \rho^{1/3}\rho_0^{2/3} = (\rho_0 - \rho)/3 + (\rho_0 - \rho)^2/9 + \ldots$. Retaining only the first term, we obtain

$$p = \frac{\rho_0 \rho v'}{\rho_0 - \rho}. \tag{9}$$

Next, consider the energy dissipation rate Φ spent in the interparticle friction. We assume, for simplicity's sake, that contribution of the fluctuation term ω'_{ji} can be neglected in comparison to the main term ω_{ji}. Then, the argument in Part 1 can be applied again. We finally have

$$\Phi = \sqrt{6}\,\mu\left(\frac{a}{r}\right) p\hat{\omega}, \tag{10}$$

where μ is the kinetic friction coefficient of particles, and $\hat{\omega}$ is the quantity defined by Eq. (18) in Part 1.

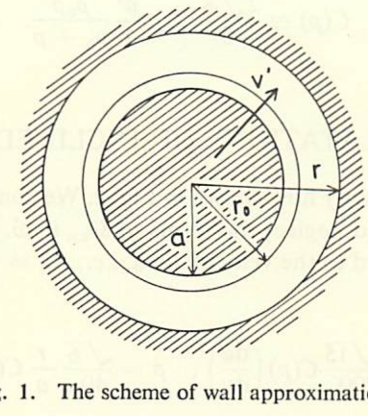

Fig. 1. The scheme of wall approximation.

Finally, we determine the "equation of state" for the "internal energy". The form $\varepsilon(v_i, \partial_j v_i, \omega_{ji}, \partial_k \omega_{ji})$ must be invariant to translations and rigid rotations of the coordinate system. Hence, we can conclude that ε is a function of quantities E_{ji}, R_{ji} and Ω_{kji} defined in Part 1. As was discussed in Part 1, $\hat{\omega}(E_{ji}, R_{ji}, \Omega_{kji})$ is the quantity representing the interparticle interactions. Hence, we can expect that ε is a function of $\hat{\omega}$. Once the form of ε is so assumed, we have only to consider the special situation in which the average velocity of the particles is zero. In this situation, the kinetic energy of rotation is $(1/5)ma^2\hat{\omega}^2$ per one particle, for $\hat{\omega}$ coincides with the magnitude of the particle rotation in this situation. The kinetic energy of the fluctuation mode is $(1/2)mv'^2$ in accordance with the scheme of wall approximation. We can expect that in "local equilibrium" the latter kinetic energy increases as the former increases. We now postulate that the total kinetic energy of the particle is partitioned into these two modes of motion in a fixed ratio in "local equilibrium". Thus, $(1/2)mv'^2 = T_e(1/5)ma^2\hat{\omega}^2$, where T_e is the proportionality constant. Hence, we have

$$\varepsilon = \frac{1}{5} T_e \rho a^2 \hat{\omega}^2, \quad v' = \frac{\sqrt{10}}{5} \sqrt{T_e} a\hat{\omega}. \tag{11), (12}$$

Substitution of this in (10) and (11) yields the following "equations of state" for the pressure and the energy dissipation rate Φ such as

$$p = \frac{2}{5} T_e a^2 \frac{\rho_0 \rho}{\rho_0 - \rho} \hat{\omega}^2, \quad \Phi = \frac{2\sqrt{6}}{5} T_e \mu \frac{a^3}{r} \frac{\rho_0 \rho}{\rho_0 - \rho} \hat{\omega}^3. \tag{13), (14}$$

The dissipative stresses are determined by the procedure discussed in Part 1 as follows (See Part 1[1]):

$$\bar{\sigma}^{ji} = \frac{3}{10} C(\rho)\hat{\omega}(\partial_{(j} v_{i)} - \frac{1}{3}\delta_{ji}\partial_k v^k), \tag{15}$$

$$\sigma^{[ji]} = \frac{1}{2} C(\rho)\hat{\omega}(\partial_{[j} v_{i]} - \omega_{ji}), \tag{16}$$

$$\mu^{kji} = \frac{1}{5} a^2 C(\rho)\hat{\omega}(\delta_{k[j}\partial_{|l|}\omega_{l|i]} + \partial_k \omega_{ji} - \partial_{[j}\omega_{i]k}), \tag{17}$$

$$C(\rho) \equiv \frac{2\sqrt{6}}{5} T_e \mu \frac{a^3}{r} \frac{\rho_0 \rho}{\rho_0 - \rho}. \tag{18}$$

IV. "THERMAL DILATATION" OF INCLINED GRAVITY FLOWS

Consider the inclined gravity flow shown in Fig. 2. We consider the case $a/h \ll 1$, where h is the depth of the flow, and neglect the terms of Ω_{kji} in $\hat{\omega}$. If the flow is steady and the particle rotation is constrained to the velocity field, *i.e.*, $R_{ji} = 0$, the shear stress σ^{yx} and the pressure p are given by

$$\sigma^{yx} = \frac{3\sqrt{15}}{200} C(\rho) \left(\frac{\mathrm{d}u}{\mathrm{d}y}\right)^2, \quad p = \frac{\sqrt{6}}{40\mu} \frac{r}{a} C(\rho) \left(\frac{\mathrm{d}u}{\mathrm{d}y}\right)^2 \tag{19}$$

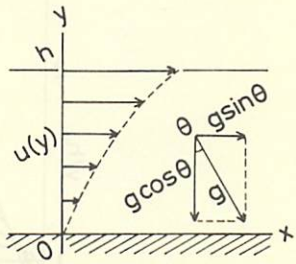

Fig. 2. Inclined gravity flow.

where u is the x-component of the velocity as is indicated in Fig.2. Both of the shear stress and the pressure are proportional to the square of the velocity gradient $(\mathrm{d}u/\mathrm{d}y)^2$ as was derived and experimentally confirmed by Bagnold[6]. If the flow is the simple shear flow, the equation of motion for the y-component of the velocity is easily integrated to give the form of $\rho(y)$. The integration constant is determined by the mass conservation relation

$$\int_0^\infty \rho \mathrm{d}y = \rho_0 h_0, \tag{20}$$

where h_0 is the depth of the slab when all the particles are at rest. The result is expressed in the following implicit form:

$$\frac{y}{h_0} = 1 + A\left(1 - \log\frac{A\rho}{\rho_0 - \rho} - \frac{\rho_0}{\rho_0 - \rho}\right), \tag{21}$$

$$A \equiv \frac{3T_e a^2}{50gh_0\cos\theta}\left(\frac{\mathrm{d}u}{\mathrm{d}y}\right)^2. \tag{22}$$

Figure 3 shows the density profile for the flow. It is seen that the increase of the shearing leads to the increase of the "internal energy", which in turn causes the "thermal dilatation" of the flow. This fact was observed by Ono[9]. Using this form of $\rho(y)$, we can calculate the acceleration profile $\partial u/\partial t$ from the equation of motion for u. If there is no slip at the bottom $y = 0$, we obtain the acceleration profile shown in Fig. 4. We can see that the acceleration is especially large in the upper layer of the flow, as was also observed by Ono[9]. If the density profile does not change rapidly, we obtain the time varying velocity profile shown in Fig. 5.

V. "ENTROPY" OF NON-EQUILIBRIUM FLOWS

Let us consider flows not in "local equilibrium". In non-equilibrium flows, the "internal energy" ε is an independent "thermodynamic" quantity depending upon the history of the motion. However, we can consider the "equation of state" (12) to be still valid, regarding T_e, instead, not as a constant but as a new independent quantity T which depends on the

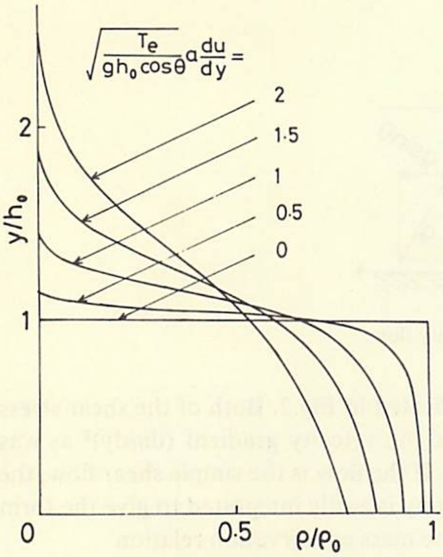

Fig. 3. Density profile for the fast flow.

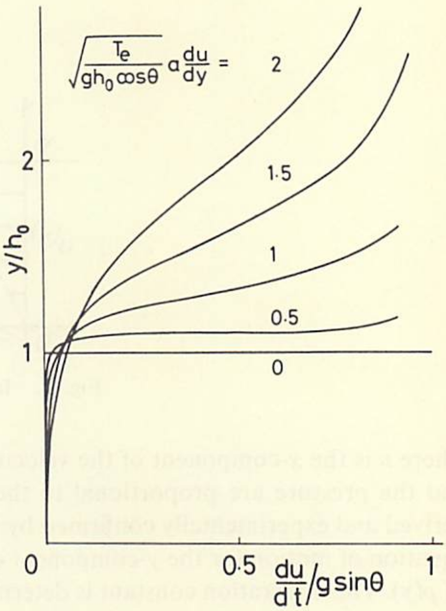

Fig. 4. Acceleration profile for the fast flow.

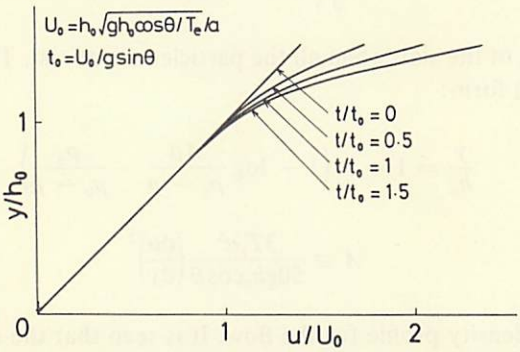

Fig. 5. Velocity profile for the fast flow.

history of motion. The "equation of state" (13) for the pressure p is rewritten in terms of the occupation volume V as

$$p(V - V_0) = \frac{2}{5}\, Tma^2\hat{\omega}^2, \tag{23}$$

where V_0 is the occupation volume of randomly packed spheres. By analogy with the equation of state for ideal gas, we can say that T plays the role of temperature. Hence, we call T the "temperature" of the flow. From this viewpoint, the interpretation of Fig. 3 is that the rise of "temperature" causes the "thermal dilatation".

Let $d\varepsilon/dt$ be partitioned into two parts such as

$$\frac{d\varepsilon}{dt} = \frac{d^e\varepsilon}{dt} + \frac{d^*\varepsilon}{dt}, \quad \rho\frac{d^e\varepsilon}{dt} = -p\partial_i v^i. \tag{24}$$

From (7), we obtain

$$\rho\frac{d^*\varepsilon}{dt} = \Phi + q - \partial_i h^i. \tag{25}$$

By definition, the non-equilibrium part $d^*\varepsilon/dt$ vanishes in "local equilibrium". In order to assure the approach to "local equilibrium", we postulate that $d^*\varepsilon/dt > 0$ for $T < T_e$ and that $d^*\varepsilon/dt < 0$ for $T > T_e$. The "heat flux" h^i is assumed to be in the direction of lower "temperature". These assumptions are written as

$$\frac{1}{T - T_e}\frac{d^*\varepsilon}{dt} < 0, \quad h^i\partial_i T < 0. \tag{26}$$

Let the "entropy" s per unit mass be defined as a quantity which obeys the following equation.

$$\rho\frac{ds}{dt} = -\frac{\Phi}{T - T_e}. \tag{27}$$

The initial value of s is irrelevant. Then, we have

Theorem

Let the "entropy production rate" η be defined by the following "entropy balance equation".

$$\frac{d}{dt}\int \rho s dV = \int \eta dV + \int \frac{q}{T - T_e} dV - \int \frac{h^i}{T - T_e} n_i dS. \tag{28}$$

Then

$$\eta > 0. \tag{29}$$

Proof

Integrating Eq. (28) by parts, we have

$$\eta = \rho\frac{ds}{dt} - \frac{q - \partial_i h^i}{T - T_e} - \frac{h^i\partial_i T}{(T - T_e)^2}. \tag{30}$$

Substitution of Eqs. (25) and (27) in this yields

$$\eta = -\frac{1}{T - T_e}\rho\frac{d^*\varepsilon}{dt} - \frac{h^i\partial_i T}{(T - T_e)^2}. \tag{31}$$

From Eq. (26) we have the theorem.

VI. CONCLUDING REMARKS

We have presented a polar continuum model for fast flows of granular materials. We have adopted notions of thermodynamics and presented a consistent description of the flow in "local equilibrium". The "thermal dilatation" of the flow is analyzed to see the *normal stress effects, i.e.*, the pressure caused by the velosity gradient of the flow. Finally, we have obtained an "entropy" formulation which is an extention of classical continuum thermodynamics.

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