The Semisimple Case; Harish-Chandra Modules

If V is a Banach space and $G \to V, g \mapsto \pi(g)v$ is real analytic, then v is called an analytic vector. For semisimple G (with finite center) pick a maximal compact subgroup K and call $v \in V$ ***K-finite*** if $\pi(K)v$ spans a finite dimensional space. For a wide class of representations (called the ***admissible*** ones), the space V_K of K-finite vectors is dense in V, consists of analytic vectors and is invariant under $d\pi(\mathfrak{g})$ as well as $\pi(K)$. The formula $d\pi(X)v = \frac{d}{dt}|_{t=0}\pi(\exp tX)v$ for $X \in \mathfrak{k}$, the Lie algebra of K, yields a compatibility condition between the $\mathfrak{g}$- and the K-action on V_K. Spaces with compatible K- and $\mathfrak{g}$-actions in which all vectors are K-finite are called ***Harish-Chandra modules***. One can show that the category of Harish-Chandra modules is equivalent to the category of admissible representations. Thus the basic questions of representation theory like classification of irreducibles and decomposition into irreducibles can be dealt with in terms of Harish-Chandra modules. This also makes plausible that in the study of admissible representations the ***K-spectrum***, i.e., the decomposition of $\pi|_K$ into irreducible subrepresentations, plays a fundamental role.

The Orbit Method

Unitary representations are the most important class of infinite dimensional representations. One reason for this is the fact that they show up as symmetry operations in quantum mechanics. It is still an open problem to classify all the irreducible unitary representations (up to the natural isomorphism of such representations). This is true even for the class of simple Lie groups. Kirillov's ***orbit method***, also known under the name ***geometric quantization***, is a very general method for the construction and classification of unitary representations which works best for simply connected nilpotent Lie groups (where it achieves the desired classification) but it provides general insights and specific results also in many other contexts. The basic idea is to construct unitary representations of G from linear functionals on $\mathfrak{g}$ via induction. It turns out that functionals which are conjugate under the ***coadjoint action***, i.e., the dual of the adjoint action, lead to isomorphic representations. For simply connected nilpotent groups one obtains in this way a bijection between the set of coadjoint orbits and isomorphism classes of unitary representations.

Unitarizability

Langlands obtained a classification of irreducible admissible representations of semisimple Lie groups via parabolic subgroups and induced representations. The *Langlands parameters*, however, give no clue whether the representation is ***unitarizable***, i.e., whether the representation space can be given an inner product such that the representation is unitary. There are important classes of representations and groups for which the unitarizability problem has been solved, but the general problem is still open.

Applications

Very often, the representation theory of Lie groups is used for *harmonic analysis* type questions: Given a natural group action on a space, describe the building blocks of the representations on function or section spaces associated with this action. Such questions occur also in number theory (cf.Langlands program), geometry and invariant theory. Important applications of finite and infinite dimensional Lie group representations can be found in quantum mechanics (e.g., classification of elementary particles). Analytic properties of representations can be used in ergodic theory and its applications to metric number theory. Even though representations of Lie groups are still studied for their own sake, one observes a recent shift of emphasis towards applications of representation theory to other fields.

More on this topic can be found in (Bröcker and tom Dieck 1985, Corwin and Greenleaf 1990, Howie 1992, Knapp 1986, Wallach 1988,1992).

E.9 Groups in 2D/3D vision problems

by Kenichi Kanatani in Okayama, Japan and Günter F. Pilz in Linz, Austria

Image understanding is a task which shows up in a vast variety of applied mathematics: automatic traffic control via a camera (how can the camera distinguish between cars and other objects?), Computer Tomography, recognition of aircrafts from their silhouettes, and so on.

All of these problems are 2D/3D vision problems: how to extract knowledge about 3D objects from 2D images. Similarly, 2D/2D problems (*pattern recognition*) turn out in automatic text readers, and the like. For the details, see (Kanatani 1990).

Suppose we have a map from a 3D-scene to a 2D-(camera) image. ***Image parameters*** are numerical data extracted from the 3D-scene (like the grey level of a point). ***Object parameters*** are *properties* of the 3D-scene, like the speed of a certain object. ***Recovery equations*** relate the (known) image parameters $c_1, \ldots, c_m$ to the (unknown) object parameters $\alpha_1, \ldots, \alpha_n$ via

$$c_1 = F_1\left(\alpha_1, \ldots, \alpha_n\right), \ldots, c_m = F_m\left(\alpha_1, \ldots, \alpha_n\right)$$

These recovery equations are usually obtained from laws of geometry, of optics, etc.; they may be implicit and hard to solve. Often free parameter turn up which have to be understood. We will see now via an example how group theory can help.

E.9.1 Example Consider a plane P with equation $z = px + qy + r$ which travels through the 3D-space in a combination of translations and rotations (Figure 1). A camera is fixed high up on the z-axis (to avoid perspectives). Here p, q, r, a, b, c, w_1, w_2, and w_3 are suitable object

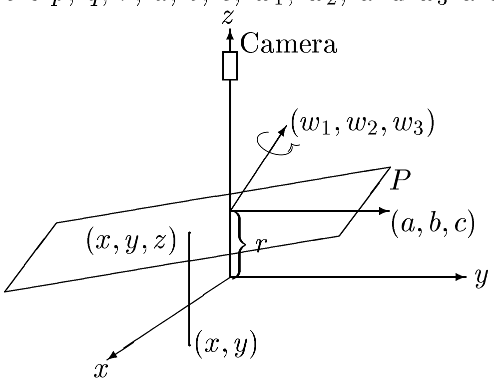


Figure 1: Translation and rotation of a plane

parameters which completely describe the journey of P. The camera measures the velocity field $\dot{x} = u\,(x,y)$, $\dot{y} = v\,(x,y)$. Physics gives us

$$\begin{pmatrix} u\,(x,y) \\ v\,(x,y) \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

with $A := pw_2$, $B := qw_2 - w_3$ $C := -pw_1 + w_3$, $D := -qw_1$. So a, b, A, B, C, D can be computed and taken as image parameters, and $A = pw_2$ etc. (together wirh $a = a$, $b = b$) constitute the recovery equations. Apart from $a = a$ and $b = b$, we thus have 4 equations for the 5 unknowns p, q, w_1, w_2, w_3; we have to expect a free parameter. Observe that r and c cannot be recovered at all.

Not every combination of image parameters is geometrically *meaningful*; a has no meaning, while (a, b) has one. Some thinking leads to the definition of *meaning.* A simple image parameter is meaningful if it does not change its value after a coordinate rotation. So the grey level of the darkest point is a meaningful parameter, while the x-coordinate of a point is not.

If $\mathbf{R}_\theta$ is the rotation of the xy-plane around the origin by angle θ, a pair (c_1, c_2) of image parameters is meaningful if they transform to (c_1', c_2') under a rotation like points, i.e., $\binom{c_1'}{c_2'} = \mathbf{R}_\theta \binom{c_1}{c_2} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \binom{c_1}{c_2}$.

Finally, (c_1, c_2, c_3, c_4) is defined to be meaningful if $\mathbf{M} := \begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix}$ transforms to $\mathbf{M}' = \begin{pmatrix} c_1' & c_2' \\ c_3' & c_4' \end{pmatrix}$ as $\mathbf{M}' = \mathbf{R}_\theta \mathbf{M} \mathbf{R}_\theta^{-1}$ for all θ.

If $(c_1, \dots, c_k)$ are image parameters, we can also take $\begin{pmatrix} C_1 \\ \vdots \\ C_k \end{pmatrix} = \mathbf{P} \begin{pmatrix} c_1 \\ \vdots \\ c_k \end{pmatrix}$, provided that $\mathbf{P}$ is regular.

Suppose we find some $\mathbf{P}$ such that the coordinate rotation induces a linear mapping in a block diagonal form $\operatorname{diag}(Q_1, \dots, Q_r)$, for all θ. Then

$$\begin{pmatrix} C_1' \\ \vdots \\ C_k' \end{pmatrix} = \operatorname{diag}\left(Q_1 \begin{pmatrix} C_1 \\ \vdots \\ C_{r_1} \end{pmatrix}, Q_2 \begin{pmatrix} C_{r_1+1} \\ \vdots \\ C_{r_2} \end{pmatrix}, \dots \right)$$

so $C_1, \dots, C_{r_1}$ transforms independently of $C_{r_1+1}, \dots, C_{r_2}$, and so on. Hence $C_1, \dots, C_{r_1}$ and $C_{r_1+1}, \dots, C_{r_2}$ and ... indicate **single** and **independent properties**.

Now we are right in representation theory. We have a representation h of the group $\mathrm{SO}(2)$ of rotations of $\mathbb{R}^2$ by the subgroup $\{ \mathbf{R}_\theta \mid 0 \leq \theta < 2\pi \}$ of all regular 2×2 matrices, and we want to decompose h into irreducible representations $h_1, \dots, h_k$ to get the irreducible components $Q_1, \dots, Q_k$ above. But this is one of the main purposes of representation theory!

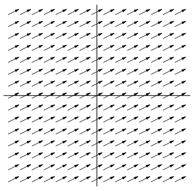
The following statements are sometimes called *Weyl's Thesis* (Hermann Weyl, 1888-1955):

1 *Reducing a representation amounts to decomposing a complex property into independent ones.*

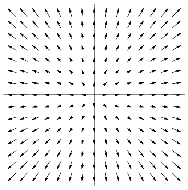
2 *Irreducible representations correspond to single properties.*

Example (revisited). In our example, (a, b) and (A, B, C, D) can be seen to be meaningful. With $\mathbf{P} := \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix} \in \mathrm{GL}(2; \mathbb{C})$, we have $\mathbf{P}\mathbf{R}_\theta\mathbf{P}^{-1} = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$ for all θ. Hence we transform (a, b) into $a + ib$ and $a - ib$. Similarly, from (A, B, C, D) we get the single properties $A + D$, $B - C$ and $(A - D) \pm i\,(B + C)$; the recovery equations can now be solved, and the free parameter just turns out to be a scaling parameter. For the details, see (see Kanatani 1990) or (Lidl and Pilz 1998).

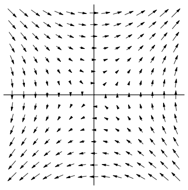
The resulting flows can also be characterized now: if $A = B = C = D = 0$, the flow is just a translation $\binom{u}{v} = \binom{a}{b}$; if only $B - C \neq 0$, we get a rotational flow; if all parameters except $A + D$ are zero, we have a divergent flow; and if only $C_3 \neq 0$, we have a shear flow.



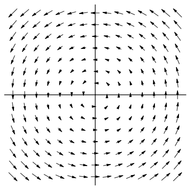
Translational flow



Divergent flow



Shear flow



Rotational flow

In summary, *reducing representations means decoupling overlapping influences!*

E.10 Representation Theory and Statistics

by Albert Fässler in Biel, Switzerland

Data sometimes come in the form of rankings or preferences, for example in marketing polls or in certain types of elections. Here we analyze a representative problem, which of course can be generalized. The American Psychological Association elects its president by asking its members to rank five candidates, denoted by A, B, C, D, E. Many members cast incomplete ballots, voting for their favorite $q = 1, 2$ or 3 out of 5 candidates. Here is a summary of some selected data of the 1980 election, with the case $q = 5$ describing the fully ranked data:

$q = 5$	$q = 1$	$q = 2$	$q = 3$
$f(12345) = 30$	$f(00001) = 1022$	$f(00012) = 196$	$f(00123) = 15$
$f(12354) = 28$	$f(00010) = 1145$	$f(00021) = 143$	$f(00132) = 19$
$f(12435) = 27$	$f(00100) = 1198$	$f(00102) = 93$	$f(01023) = 55$
$\ldots$	$f(01000) = 881$	$\ldots$	$\ldots$
$f(54321) = 29$	$f(10000) = 895$	$f(21000) = 72$	$f(32100) = 57$
$\sum f(x) = 5738$	$\sum f(x) = 5141$	$\sum f(x) = 2462$	$\sum f(x) = 2108$
$\dim f = 120$	$\dim f = 5$	$\dim f = 20$	$\dim f = 60$
$f(x) \in [11, 172]$	$f(x) \in [881, 1198]$	$f(x) \in [48, 547]$	$f(x) \in [8, 79]$

Interpretations: $f(54321) = 29$ means that 29 members voted for candidate E in first, candidate D in second position, ... Zeros indicate unranked data: $f(00102) = 93$ means that 93 members ranked candidate C in first and candidate E in second position. In every column, the data vector is called f, even though they are all different. The symbol x stands for rankings.

A first order analysis of the fully ranked data ($q = 5$) showed that candidate C received the highest number of first position votes, while candidate A was "lowest" in average rank, due to few "hate votes". Now, should A or C be the president?

In order to make an adequate decision, further analysis is needed. The following group theoretical investigation will show that there is more structure in the data as one might think, allowing for better statistical interpretation.

Let $\lambda = (\lambda_1, \lambda_2, \dots \lambda_r)$ be a partition of n with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r$.

A ***ranking of shape*** λ is specified as follows: Choose your favorite λ_1 candidates without internal ranking, then choose your next λ_2 favorite candidates, again without internal ranking, and so on.

For example, data based on people's favorite and least favorite item are indexed by $(1, n{-}2, 1)$, this is equivalent to $(n{-}2, 1, 1)$. The Partition $(1,1,1,1,1)$ means complete rankings, $(3,1,1)$ means a choice of ordered subsets 2 out of 5. Each permutation $\pi \in S_n$ transforms the set R^λ of the $n!/(\lambda_1!\lambda_2!\cdots\lambda_r!)$ rankings of shape λ.

Here are the transforms for a ranking of shape $(3,2)$ of the election problem:

$$\pi\begin{pmatrix} \underline{A} & \underline{B} & \underline{E} \\ \underline{C} & \underline{D} & \end{pmatrix} = \begin{matrix} \underline{\pi(A)} & \underline{\pi(B)} & \underline{\pi(E)} \\ \underline{\pi(C)} & \underline{\pi(D)} & \end{matrix} \tag{1}$$

In the argument of π, candidates A, B, E are ranked first and C, D are ranked second. Underlining denotes unordered rows.

Let us consider the representations $\vartheta^\lambda \colon \pi \to D^\lambda(\pi)$ of S_n:

$$D^\lambda(\pi)\varphi(x) = \varphi(\pi^{-1}(x)), \quad \text{where } \varphi \text{ is a real function on } R^\lambda. \tag{2}$$

The table below shows the ***spectral analysis*** of the four data vectors: The corresponding representations ϑ^λ have irreducible parts ϑ_μ, labeled by the ordered partitions μ of 5. Their multiplicities and dimensions can be calculated by use of Young's rule and the hooklength formula (see Section E.7). SS means the square sums of the different isotypic components f_μ of the data vectors f, where the f_μ's can be obtained with the projection in Theorem E.4.3.