CVGIP: IMAGE UNDERSTANDING
Vol. 58, No. 3, November, pp. 286–301, 1993

3D Interpretation of Conics and Orthogonality

KENICHI KANATANI AND WU LIU

Department of Computer Science, Gunma University, Kiryu, Gunma 376, Japan

Received October 7, 1991; accepted October 22, 1992

Computational techniques involving conics are formulated in the framework of projective geometry, and basic notions of projective geometry such as poles, polars, and conjugate pairs are reformulated as "computational procedures" with special emphasis on computational aspects. It is shown that the 3D geometry of three orthogonal lines can be interpreted by computing conics. We then describe an analytical procedure for computing the 3D geometry of a conic of a known shape from its projection. Real image examples are also given. © 1993 Academic Press, Inc.

1. INTRODUCTION

"Conics" provide the most important clues to 3D interpretation of images next to straight lines. One reason for this is that many man-made objects have circular or spherical parts, and circles and spheres are projected onto conics. Another reason is that conics are very easy to handle because they are the lowest-degree algebraic curves other than straight lines, and many curves can be approximated by conics. In addition, conics are invariant to projective transformations, which include perspective transformations. Thus, the study of conics is vital to developing computer vision systems, as has been widely recognized by many researchers, e.g., [2].

Since the conic is one of the central topics of projective geometry, its mathematical properties have been well known, as can be found in any textbook on projective geometry, e.g., [12]. It appears, therefore that image analysis of conics is readily done by simply referring to such textbooks. However, a considerable gap exists between treating conics as mathematical entities and actually analyzing images of conics, because the aim of projective geometry is generalization and abstraction. In other words, littel attention is paid to "practical" (or "computational") issues. Put differently, projective geometry can be regarded as a mature branch of mathematics for the very reason that it is no longer concerned with the "real" world.

This paper attempts to *reformulate* those properties of conics which are vital to computer vision systems as *computational procedures*. Following Kanatani [9], we call the resulting theory *computational projective geometry*. We first describe fundamental notions of projective geometry such as *collineations*, *poles*, *polars*, and *conjugate pairs* as computational procedures. Then, we discuss the problem of interpreting the 3D geometry of three orthogonal lines in the scene. This is a very important issue of computer vision because many man-made objects have rectangular corners. This problem has already been solved by several authors [1, 3, 4, 6, 8, 13]. However, our formulation provides a good example of demonstrating how otherwise complicated procedures can be succinctly and elegantly described in the framework of computational projective geometry.

Then, we analyze the problem of interpreting the 3D geometry of a conic in the scene. Two cases are considered: the case where the conic is known to be a projection of a circle of a known shape, and the case where the conic is known to be a projection of an ellipse of a known shape. The solution in the first case has been known and used in real systems [2]. Our approach is the same in both cases: we apply a special "collineation" called the *camera rotation transformation* [5–8]. Real image examples are also given to observe the accuracy of the computation.

2. N-VECTORS AND COLLINEATIONS

Assume the following camera imaging model. The camera is associated with an XYZ coordinate system with origin O at the center of the lens and Z-axis along the optical axis (Fig. 1). The plane $Z = f$ is identified with the image plane, on which an xy image coordinate system is defined so that the x- and y-axes are parallel to the X- and Y-axes, respectively. Let us call the origin O the *viewpoint* and the constant f the *focal length*.

A point (x, y) on the image plane is represented by the unit vector $\mathbf{m}$ indicating the orientation of the ray starting from the viewpoint O and passing through that point; a line $Ax + By + C = 0$ on the image plane is represented by the unit surface normal $\mathbf{n}$ to the plane passing through the viewpoint O and intersecting the image plane along that line (Fig. 1). Their components are

1049-9660/93 $5.00
Copyright © 1993 by Academic Press, Inc.
All rights of reproduction in any form reserved.

$$\mathbf{m} = \pm N\left[\begin{pmatrix} x \\ y \\ f \end{pmatrix}\right], \quad \mathbf{n} = \pm N\left[\begin{pmatrix} A \\ B \\ C/f \end{pmatrix}\right], \tag{1}$$

where $N[\cdot]$ denotes the normalization into a unit vector. We call $\mathbf{m}$ and $\mathbf{n}$ the *N-vectors* of the point and the line [9]. If $\mathbf{m}$ and $\mathbf{n}$ are the N-vectors of a point P and a line l, respectively, point P is *on* line l, or line l *passes through* point P, if and only if

$$(\mathbf{m}, \mathbf{n}) = 0, \tag{2}$$

where $(\cdot, \cdot)$ denotes the inner product of vectors. If this is satisfied, we also say that point P and line l are *incident* to each other and we call Eq. (2) the *incidence equation*.

The use of N-vectors for representing points and lines on the image plane is equivalent to using *homogeneous coordinates* [12]. Although homogeneous coordinates can be multiplied by any nonzero number, computational problems arise if they are too large or too small. So, it is convenient to normalize the three components into a unit vector, which is precisely the N-vector as defined above. Kanatani [9] reformulated projective geometry from this viewpoint. Rewriting the relationship of projective geometry as *computational procedures*, he called the resulting formalism *computational projective geometry*. In this paper, we adopt his formalism, regarding a unit vector $\mathbf{m}$ whose Z-component is 0 as the N-vector of an *ideal point* (a *point of infinity*) and $\mathbf{n} = (0, 0, \pm 1)$ as the N-vector of the *ideal line* (the *line at infinity*).

Points are *collinear* if they are all on a common line; lines are *concurrent* if they all meet at a common point. A *collineation* is a one-to-one mapping between points (including ideal points) and between lines (including the ideal line) such that (i) collinear points are mapped to collinear points, (ii) concurrent lines are mapped to concurrent lines, and (iii) incidence is preserved—if a point (or line) is on (or passes through) a line (or point), the mapped point (or line) is on (or passes through) the mapped line (or point). It can be proved that a collineation maps a point of N-vector $\mathbf{m}$ to a point of N-vector $\mathbf{m}'$, and a line of N-vector $\mathbf{n}$ to a line of N-vector $\mathbf{n}'$, in the form

$$\mathbf{m}' = \pm N[\mathbf{A}^T\mathbf{m}], \quad \mathbf{n}' = \pm N[\mathbf{A}^{-1}\mathbf{n}], \tag{3}$$

where $\mathbf{A}$ is a nonsingular matrix and T denotes transpose (see [10] for details). In order to eliminate the scale indeterminacy, we hereafter adopt the scaling $\det \mathbf{A} = 1$. For simplicity, let us call the collineation represented by matrix $\mathbf{A}$ simply "collineation $\mathbf{A}$." In *inhomogeneous coordinates* (i.e., image coordinates), the first of Eqs. (3) for $\mathbf{A} = (A_{ij})$, $i, j = 1, 2, 3$, is rewritten as

$$x' = f\frac{A_{11}x + A_{21}y + A_{31}f}{A_{13}x + A_{23}y + A_{33}f}, \quad y' = f\frac{A_{12}x + A_{22}y + A_{32}f}{A_{13}x + A_{23}y + A_{33}f}. \tag{4}$$

As can be seen from Eqs. (3), the mapping rule for N-vectors of points is different from that for N-vectors of lines. This is a consequence of the requirement that incidence be preserved: N-vectors $\mathbf{m}$ and $\mathbf{n}$ such that $(\mathbf{m}, \mathbf{n}) = 0$ must be mapped to N-vectors $\mathbf{m}'$ and $\mathbf{n}'$ such that $(\mathbf{m}', \mathbf{n}') = 0$. This fact is expressed by saying that the mapping of points and the mapping of lines are *contragradient* to each other: a vector mapped as an N-vector of a point is called a *contravariant vector*; a vector mapped as an N-vector of a line is called a *covariant vector* [12]. The set of all collineations is the *group of 2D projective transformations*, which is isomorphic to $SL(3)$—the group of three-dimensional matrices of determinant 1 under matrix multiplication. A collineation is also called a *projective transformation* or simply *projectivity*.

The group of 2D projective transformations contains many subgroups. Among them, the one which plays a fundamental role in 3D interpretation of conics is the *group of camera rotation transformations* [5–8]. Consider a point P in the scene. Let $\mathbf{m}$ be its N-vector. If the camera is rotated around the center of the lens, a new image is observed. Let $\mathbf{m}'$ be the N-vector of the same point P after the camera rotation (Fig. 2). The mapping from $\mathbf{m}$ to $\mathbf{m}'$ must be a collineation, because (i) collinear points are mapped to collinear points, (ii) concurrent lines are mapped to concurrent lines, and (iii) incidence is preserved. In fact, if the camera rotation is specified by rotation matrix $\mathbf{R}$, this collineation is given by

$$\mathbf{m}' = \pm\mathbf{R}^T\mathbf{m}, \quad \mathbf{n}' = \pm\mathbf{R}^T\mathbf{n}, \tag{5}$$

because rotating the camera relative to the scene by $\mathbf{R}$ is equivalent to rotating the scene relative to the camera by

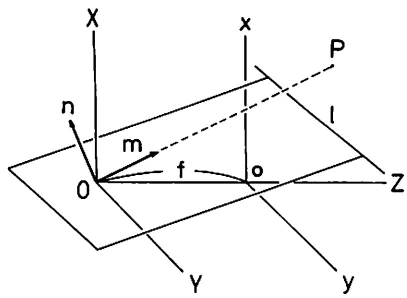

FIG. 1. Camera imaging geometry and N-vectors of a point and a line.

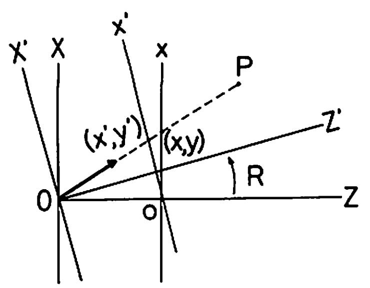

FIG. 2. Rotating the camera relative to the scene by $\mathbf{R}$ is equivalent to rotating the scene relative to the camera by $\mathbf{R}^{-1}$ ($=\mathbf{R}^T$).

$\mathbf{R}^{-1}$ ($=\mathbf{R}^T$) (rotations do not change the norms of vectors, so the normalization $N[\cdot]$ is not necessary; also det $\mathbf{R}$ = 1). In image coordinates, Eq. (5) for $\mathbf{R} = (R_{ij})$, $i, j = 1, 2, 3$, is rewritten as

$$x' = f\frac{R_{11}x + R_{21}y + R_{31}f}{R_{13}x + R_{23}y + R_{33}f},\quad y' = f\frac{R_{12}x + R_{22}y + R_{32}f}{R_{13}x + R_{23}y + R_{33}f}. \tag{6}$$

The set of all camera rotation transformations is a subgroup of the group of 2D projective transformations.

Another familiar subgroup is the *group of 2D Euclidean motions* generated by rotations and translations of the image plane in the form

$$x' = x\cos\theta - y\sin\theta + a,\quad y' = x\sin\theta + y\cos\theta + b. \tag{7}$$

In matrix form,

$$\mathbf{A} = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ a/f & b/f & 1 \end{pmatrix}. \tag{8}$$

3. FUNDAMENTAL PROPERTIES OF CONICS

A quadratic curve on the image plane has the form

$$Ax^2 + 2Bxy + Cy^2 + 2(Dx + Ey) + F = 0. \tag{9}$$

In terms of N-vector $\mathbf{m}$, this equation is written as

$$(\mathbf{m}, \mathbf{Qm}) = 0,\quad \mathbf{Q} = \kappa\begin{pmatrix} A & B & D/f \\ B & C & E/f \\ D/f & E/f & F/f^2 \end{pmatrix}, \tag{10}$$

where κ is an arbitrary nonzero constant. The set of N-vectors defined in this form for an arbitrary nonzero real symmetric matrix $\mathbf{Q}$ is called a *conic* (it does not always define a curve on the image plane). For brevity, we call it simply "conic $\mathbf{Q}$." In order to eliminate the scale indeterminacy, we hereafter adopt the scaling det $\mathbf{Q} = -1$ (this choice is natural for real conics, as we will see shortly) whenever det $\mathbf{Q} \neq 0$.

Conics are always projected onto conics by not only perspective projections but also general collineations. The following is the transformation rule of conics under a collineation [12].

PROPOSITION 1. *Collineation* $\mathbf{A}$ *maps conic* $\mathbf{Q}$ *to conic* $\mathbf{Q}'$ *in the form*

$$\mathbf{Q}' = \mathbf{A}^{-1}\mathbf{Q}(\mathbf{A}^{-1})^T. \tag{11}$$

Let us say that a collineation that maps a conic $\mathbf{Q}$ to itself *preserves* the conic $\mathbf{Q}$. The following are immediate consequences of Proposition 1.

PROPOSITION 2. *A collineation* $\mathbf{A}$ *preserves a conic* $\mathbf{Q}$ *if and only if*

$$\mathbf{AQA}^T = \mathbf{Q}. \tag{12}$$

PROPOSITION 3. *The set of all collineations that preserve a conic is a subgroup of the group of 2D projective transformations.*

It turns out that the group of camera rotation transformations defined in the preceding section is characterized as "the subgroup of the group of 2D projective transformations that preserves the *absolute conic* $x^2 + y^2 + f^2 = 0$ (conic $-\mathbf{I}$)." Although it does not define a real curve, this absolute conic plays the fundamental role of interpreting orthogonality in the scene [9, 10].

A conic is *proper* if it does not consist of two (real or imaginary) lines or one degenerate (real or imaginary) line (including the ideal line), or equivalently if Eq. (9) is irreducible in the complex domain. It can be proved [12] that

PROPOSITION 4. *A conic* $\mathbf{Q}$ *is proper if and only if the corresponding matrix* $\mathbf{Q}$ *is nonsingular.*

A proper conic does not necessarily define a real curve (an ellipse, parabola, or hyperbola) on the image plane. It can be proved [12] that

PROPOSITION 5. *A proper conic* $\mathbf{Q}$ *is a real conic if and only if its signature is* (2, 1).

The *signature* of a symmetric matrix is a pair (p, q) of integers, where p is the number of its positive eigenvalues and q is the number of its negative eigenvalues. A real proper conic can be transformed into its *canonical form* by applying an appropriate 2D Euclidean motion (Eqs. (7) and (8)). It is classified as an *ellipse*, *hyperbola*, or *parabola* as follows:

THEOREM 1. *Let*

$$\mathbf{Q} = \begin{pmatrix} A & B & D/f \\ B & C & E/f \\ D/f & E/f & F/f^2 \end{pmatrix}, \tag{13}$$

be a proper conic. Let

$$\lambda_1, \lambda_2 = \frac{(A + C) \pm \sqrt{(A + C)^2 - 4(AC - B^2)}}{2}: \tag{14}$$

1. *If $AC - B^2 \neq 0$, then λ_1 and λ_2 are both nonzero. Define*

$$\mu = \frac{f^2}{AC - B^2}. \tag{15}$$

$$a = \sqrt{|\mu/\lambda_1|}, \quad b = \sqrt{|\mu/\lambda_2|}. \tag{16}$$

(a) *If $\mu\lambda_1 < 0$ and $\mu\lambda_2 < 0$, then conic $\mathbf{Q}$ is imaginary.*

(b) *If $\mu\lambda_1 > 0$ and $\mu\lambda_2 > 0$, then conic $\mathbf{Q}$ is an ellipse. Its canonical form is*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. \tag{17}$$

(c) *If $\mu\lambda_1$ and $\mu\lambda_2$ have different signs, conic $\mathbf{Q}$ is a hyperbola. Its canonical form is*

$$\pm \frac{x^2}{a^2} \mp \frac{y^2}{b^2} = 1, \tag{18}$$

where the upper signs correspond to the case of $\mu\lambda_1 > 0$ and $\mu\lambda_2 < 0$ while the lower signs correspond to the case of $\mu\lambda_1 < 0$ and $\mu\lambda_2 > 0$.

2. *If $AC - B^2 = 0$, then either λ_1 or λ_2 is 0, and conic $\mathbf{Q}$ is a parabola:*

(a) *If $\lambda_1 \neq 0$ and $\lambda_2 = 0$, its canonical form is*

$$y = \left| \frac{(A + C)\sqrt{A^2 + B^2}}{2(BD - AE)} \right| x^2. \tag{19}$$

(b) *If $\lambda_1 = 0$ and $\lambda_2 \neq 0$, its canonical form is*

$$y = \left| \frac{(A + C)\sqrt{B^2 + C^2}}{2(BE - CD)} \right| x^2. \tag{20}$$

COROLLARY 1. *Let*

$$\mathbf{Q} = \begin{pmatrix} A & B & D/f \\ B & C & E/f \\ D/f & E/f & F/f^2 \end{pmatrix}, \tag{21}$$

be a proper conic:

1. *If $AC - B^2 > 0$, then*
   (a) *if $A + C > 0$, it is an ellipse;*
   (b) *if $A + C < 0$, it is an imaginary conic.*
2. *If $AC - B^2 < 0$, it is a hyperbola.*
3. *If $AC - B^2 = 0$, it is a parabola.*

4. POLARITY OF A CONIC

Let $\mathbf{Q}$ be a proper conic. If $\mathbf{m}$ is the N-vector of a point P, the line of N-vector $\mathbf{n} = \pm N[\mathbf{Qm}]$ is said to be the *polar* of P with respect to conic $\mathbf{Q}$; if $\mathbf{n}$ is the N-vector of a line l, the point P of N-vector $\mathbf{m} = \pm N[\mathbf{Q}^{-1}\mathbf{n}]$ is said to be the *pole* of line l with respect to conic $\mathbf{Q}$. In the following, we omit the proviso "with respect to conic $\mathbf{Q}$" whenever the underlying conic is understood. The following facts are well known [12].

PROPOSITION 6. *If a point P is on conic $\mathbf{Q}$, its polar is tangent to conic $\mathbf{Q}$ at P.*

COROLLARY 2. *A line is tangent to conic $\mathbf{Q}$ if and only if its pole is on conic $\mathbf{Q}$.*

PROPOSITION 7. *If a collineation maps conic $\mathbf{Q}$ to conic $\mathbf{Q}'$, and point P to point P', the polar of P with respect to $\mathbf{Q}$ is mapped to the polar of P' with respect to $\mathbf{Q}'$.*

PROPOSITION 8. *If a collineation maps conic $\mathbf{Q}$ to conic $\mathbf{Q}'$, and line l to line l', the pole of l with respect to $\mathbf{Q}$ is mapped to the pole of l' with respect to $\mathbf{Q}'$.*

Propositions 7 and 8 state that the pole–polar relationship, or *polarity*, is invariant to collineations.

The following fact is easy to observe for an ellipse: If a point is inside of it, its polar does not have (real) intersections with the conic, while if it is outside of it, its polar has two (real) intersections with the conic. This can be generalized for a general conic: A point P is defined to be *outside* (or *inside*) conic $\mathbf{Q}$ if its polar l with respect to $\mathbf{Q}$ has real (or imaginary) intersections. It can be proved [12] that (see Fig. 3):

PROPOSITION 9. *If point P is outside conic $\mathbf{Q}$, and if P_a and P_b are the tangent points of the two tangents to $\mathbf{Q}$ passing through P, the polar of P passes through P_a and P_b.*

Points P_u and P_v are said to be *conjugate* to each other with respect to conic $\mathbf{Q}$ if P_v is on the polar of P_u and P_u is on the polar of P_v. If $\mathbf{u}$ and $\mathbf{v}$ are their N-vectors, they are conjugate to each other if and only if

$$(\mathbf{u}, \mathbf{Qv}) = 0. \tag{22}$$

Hence, a point is *self-conjugate* (conjugate to itself) if and only if it is on conic $\mathbf{Q}$.

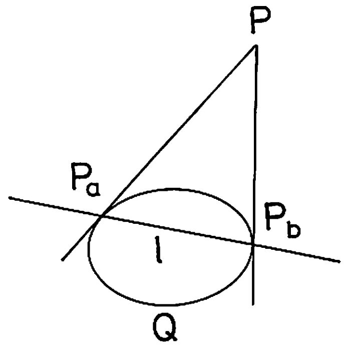

FIG. 3. The polar of a point is defined by the two tangents to the conic.

Lines l_s and l_t are said to be *conjugate* to each other with respect to conic $\mathbf{Q}$ if l_t passes through the pole of l_s and l_s passes through the pole of l_t. If $\mathbf{s}$ and $\mathbf{t}$ are their N-vectors, they are conjugate to each other if and only if

$$(\mathbf{s}, \mathbf{Q}^{-1}\mathbf{t}) = 0. \tag{23}$$

From Proposition 6 and Corollary 2, the pole of a line l is incident to l if and only if l is tangent to conic $\mathbf{Q}$. Hence, a line is *self-conjugate* if and only if it is tangent to conic $\mathbf{Q}$.

As before, we omit the proviso "with respect to conic $\mathbf{Q}$" if the underlying conic is understood.

PROPOSITION 10. *Let l be a line of N-vector $\mathbf{n}$. If point P_u of N-vector $\mathbf{u}$ is on l, its conjugate P_v on l has N-vector*

$$\mathbf{v} = \pm N[\mathbf{n} \times \mathbf{Q}\mathbf{u}] \tag{24}$$

if $N[\mathbf{Qu}] \neq \pm\mathbf{n}$. If $N[\mathbf{Qu}] = \pm\mathbf{n}$, then $\mathbf{v} = \pm\mathbf{u}$.

Proof. Since point P_v is on line l, the incidence equation $(\mathbf{v}, \mathbf{n}) = 0$ is satisfied. By definition, the N-vector of the polar of P_u is $\pm N[\mathbf{Qu}]$. Since P_v is on this polar, the incidence equation $(\mathbf{v}, \mathbf{Qn}) = 0$ is satisfied. Thus, $\mathbf{v}$ is orthogonal to both $\mathbf{n}$ and $\mathbf{Qu}$, and hence we obtain Eq. (24) if $N[\mathbf{Qu}] \neq \pm\mathbf{n}$. If $N[\mathbf{Qu}] = \pm\mathbf{n}$, P_u is self-conjugate. ∎

PROPOSITION 11. *Let P be a point of N-vector $\mathbf{m}$. If line l_s of N-vector $\mathbf{s}$ passes through P, its conjugate l_t that passes through P has N-vector*

$$\mathbf{t} = \pm N[\mathbf{m} \times \mathbf{Q}^{-1}\mathbf{s}] \tag{25}$$

if $N[\mathbf{Q}^{-1}\mathbf{s}] \neq \pm\mathbf{m}$. If $N[\mathbf{Q}^{-1}\mathbf{s}] = \pm\mathbf{m}$, then $\mathbf{t} = \pm\mathbf{s}$.

Proof. Since line l_s passes through point P, the incidence equation $(\mathbf{m}, \mathbf{s}) = 0$ is satisfied. By definition, the N-vector of the pole of l_s is $\pm N[\mathbf{Q}^{-1}\mathbf{s}]$. Since l_t passes through this pole, the incidence equation $(\mathbf{t}, \mathbf{Q}^{-1}\mathbf{s}) = 0$ is satisfied. Thus, $\mathbf{t}$ is orthogonal to both $\mathbf{m}$ and $\mathbf{Q}^{-1}\mathbf{s}$, and hence we obtain Eq. (24) if $N[\mathbf{Q}^{-1}\mathbf{s}] \neq \pm\mathbf{m}$. If $N[\mathbf{Q}^{-1}\mathbf{s}] = \pm\mathbf{m}$, l_s is self-conjugate. ∎

5. INTERSECTION WITH A LINE

Let $\mathbf{Q}$ be a proper conic. Computing the intersections of a given line l with a given conic $\mathbf{Q}$ is one of the most basic operations. The N-vectors of the intersections are easily given in terms of a "conjugate pair" of points on l.

PROPOSITION 12. *Let $\{P_u, P_v\}$ be a conjugate pair of distinct points on line l, and $\mathbf{u}$ and $\mathbf{v}$ their N-vectors. If line l has real intersections with conic $\mathbf{Q}$, the N-vectors $\mathbf{m}_1$ and $\mathbf{m}_2$ of the intersections are given by*

$$\mathbf{m}_{1,2} = \pm N[\sqrt{|(\mathbf{v}, \mathbf{Qv})|}\,\mathbf{u} \pm \sqrt{|(\mathbf{u}, \mathbf{Qu})|}\,\mathbf{v}], \tag{26}$$

where the two double signs are independent.

Proof. Since P_u and P_v are distinct, the N-vector of an arbitrary point on l is expressed in the form

$$\mathbf{m} = a\mathbf{u} + b\mathbf{v} \tag{27}$$

for some constants a and b. This point is on conic $\mathbf{Q}$ if and only if $(\mathbf{m}, \mathbf{Qm}) = 0$. Since points $\{P_u, P_v\}$ is a conjugate pair, we have $(\mathbf{u}, \mathbf{Qv}) = 0$. Hence,

$$(\mathbf{m}, \mathbf{Qm}) = a^2(\mathbf{u}, \mathbf{Qu}) + b^2(\mathbf{v}, \mathbf{Qv}) = 0. \tag{28}$$

Thus,

$$\frac{a}{b} = \pm\sqrt{-(\mathbf{v}, \mathbf{Qv})/(\mathbf{u}, \mathbf{Qu})}. \tag{29}$$

Since $\mathbf{m}$ is a unit vector, we obtain Eq. (26). ∎

If P_u is self-conjugate, it is on conic $\mathbf{Q}$ and its N-vector $\mathbf{u}$ satisfies $(\mathbf{u}, \mathbf{Qu}) = 0$. Thus, we obtain

COROLLARY 3. *Let $\{P_u, P_v\}$ be a conjugate pair on a line l, and $\mathbf{u}$ and $\mathbf{v}$ their N-vectors. The line l has real intersections with conic $\mathbf{Q}$ if and only if*

$$(\mathbf{u}, \mathbf{Qu})(\mathbf{v}, \mathbf{Qv}) \leq 0. \tag{30}$$

COROLLARY 4. *If $\{P_u, P_v\}$ is a conjugate pair on a line l that has two distinct real intersections with conic $\mathbf{Q}$, one of them is inside $\mathbf{Q}$ and the other is outside $\mathbf{Q}$.*

Proof. The conic $\mathbf{Q}$ defines image curve $(\mathbf{m}, \mathbf{Qm}) = 0$. Corollary 3 implies that $(\mathbf{u}, \mathbf{Qu})$ and $(\mathbf{v}, \mathbf{Qv})$ have different signs. Hence, P_u and P_v are on opposite sides of it. ∎

The choice of the conjugate pair $\{P_u, P_v\}$ is arbitrary, but it is convenient to choose P_u *at infinity*. The N-vector of the ideal line is $\pm\mathbf{k}$, where $\mathbf{k} = (0, 0, 1)^T$. For a line l of N-vector $\mathbf{n}$, the N-vector of its ideal point is $\mathbf{u} = \pm N[\mathbf{n} \times \mathbf{k}]$, since $\mathbf{u}$ must be orthogonal to both $\mathbf{n}$ and the Z-axis; $\mathbf{u}$ is simply the unit vector along l.

In summary, the procedure that returns the N-vectors $\mathbf{m}_1$ and $\mathbf{m}_2$ of the two real intersections (if they exist) of conic **Q** with a line of N-vector **n** is given as follows:

PROCEDURE intersection(**Q**, **n**).

1. Compute $\mathbf{u} = N[\mathbf{n} \times \mathbf{k}]$.
2. If $N[\mathbf{Qu}] = \pm\mathbf{n}$, go to *Exception* 1. Else, compute $\mathbf{v} = N[\mathbf{n} \times \mathbf{Qu}]$. If $\mathbf{v} = \pm\mathbf{u}$, go to *Exception* 2. Else,
3. If $(\mathbf{u}, \mathbf{Qu})(\mathbf{v}, \mathbf{Qv}) > 0$, there exist no (real) intersections. Else,
4. Return $\mathbf{m}_{1,2} = N[\sqrt{|(\mathbf{v}, \mathbf{Qv})|}\mathbf{u} \pm \sqrt{|(\mathbf{u}, \mathbf{Qu})|}\mathbf{v}]$.

Exception 1. This occurs if and only if the conic **Q** is a hyperbola and the line l is one of its asymptotes. So, the returned value is simply **u**.

Exception 2. This occurs if and only if the conic **Q** is a parabola and the line l is parallel to its axis. So, **u** itself is one solution. In order to obtain the other solution, let $\mathbf{u} \leftarrow \mathbf{u} + \varepsilon\mathbf{n} \times \mathbf{u}$ for an arbitrary nonzero ε and go back to Step 2.

6. 3D INTERPRETATION OF ORTHOGONAL LINES

Many man-made objects have rectangular corners. Hence, orthogonality is one of the most important clues in 3D object recognition. Aside from their frequency of occurrences, the importance of orthogonality clues lies in the fact that *their 3D geometry can be reconstructed from a projection*. This problem has already been solved by several authors by different methods based on analytical expressions in image coordinates [1, 3, 4, 6, 8, 13]. Although nothing new is gained from a "practical" point of view, we reformulate this problem in terms of N-vectors and conics; the purpose is to demonstrate *how complicated procedures in image coordinates can be succinctly and elegantly described in the framework of computational projective geometry.*

Consider three line segments l, l', and l'' on the image plane that are known to be projections of three mutually orthogonal lines in the scene. They need not have a common intersection (Fig. 4). The 3D orientation of a line in the scene is simply the N-vector of its *vanishing point* [9]. Hence, reconstructiong the 3D geometry reduces to locating the vanishing points of the three lines.

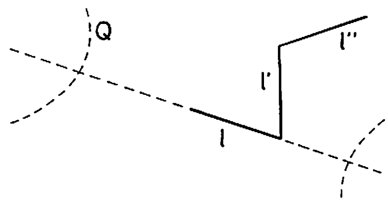

FIG. 4. The 3D geometry of three orthogonal lines in the scene can be reconstructed from their projections.

THEOREM 2. *Let l, l', and l'' be projections of mutually orthogonal lines in the scene, and* **n**, **n**′, *and* **n**″ *their respective N-vectors. The vanishing point of l is at its intersection with the conic*

$$\mathbf{Q} = (\mathbf{n}', \mathbf{n}'')\mathbf{I} - \tfrac{1}{2}(\mathbf{n}'\mathbf{n}''^{T} + \mathbf{n}''\mathbf{n}'^{T}). \tag{31}$$

Note. This conic is not normalized to $\det \mathbf{Q} = -1$: in fact $\det \mathbf{Q} = -(\mathbf{n}', \mathbf{n}'')(1 - (\mathbf{n}', \mathbf{n}'')^2)/4$.

Proof. Let **m**, **m**′, and **m**″ be the N-vectors of the vanishing points of lines l, l', and l'', respectively. Since the vanishing points are on the corresponding lines, we have the incidence equations

$$(\mathbf{m}, \mathbf{n}) = 0, \quad (\mathbf{m}', \mathbf{n}') = 0, \quad (\mathbf{m}'', \mathbf{n}'') = 0. \tag{32}$$

Since the three lines are mutually orthogonal in the scene, we have

$$(\mathbf{m}, \mathbf{m}') = 0, \quad (\mathbf{m}, \mathbf{m}'') = 0, \quad (\mathbf{m}', \mathbf{m}'') = 0. \tag{33}$$

Vector **m**′ is orthogonal to both **n**′ and **m**, and vector **m**″ is orthogonal to both **n**″ and **m**. Hence,

$$\begin{aligned} \mathbf{m}' &= \pm N[\mathbf{n}' \times \mathbf{m}] = \pm\gamma'\mathbf{n}' \times \mathbf{m}, \\ \mathbf{m}'' &= \pm N[\mathbf{n}'' \times \mathbf{m}] = \pm\gamma''\mathbf{n}'' \times \mathbf{m}, \end{aligned} \tag{34}$$

where γ' and γ'' are normalization constants. Since **m**′ and **m**″ are mutually orthogonal, we have

$$\begin{aligned} (\mathbf{m}', \mathbf{m}'') &= \pm\gamma'\gamma''(\mathbf{n}' \times \mathbf{m}, \mathbf{n}'' \times \mathbf{m}) \\ &= \pm\gamma'\gamma''((\mathbf{n}', \mathbf{n}'')(\mathbf{m}, \mathbf{m}) - (\mathbf{n}', \mathbf{m})(\mathbf{m}, \mathbf{n}'')) \\ &= \pm\gamma'\gamma''(\mathbf{m}, ((\mathbf{n}', \mathbf{n}'')\mathbf{I} - \tfrac{1}{2}(\mathbf{n}'\mathbf{n}''^{T} + \mathbf{n}''\mathbf{n}'^{T}))\mathbf{m}) \\ &= 0. \end{aligned} \tag{35}$$

Thus, vector **m** satisfies $(\mathbf{m}, \mathbf{Qm}) = 0$ if conic **Q** is defined by Eqs. (31). ■

From this, we obtain the following procedure for computing the 3D orientations **m**, **m**′, and **m**″ of three orthogonal lines in the scene that are projected onto lines of respective N-vectors **n**, **n**′, and **n**″.

PROCEDURE orthogonal(**n**, **n**′, **n**″).

1. By applying procedure intersection(**Q**, **n**), test if the line of N-vector **n** has real intersections with conic **Q** of Eqs. (31).
2. If it does not have real intersections, the image cannot be interpreted as a projection of three orthogonal lines in the scene. Else, compute the N-vector **m** of the intersection by applying procedure intersection(**Q**, **n**).

3. Compute $\mathbf{m}' = N[\mathbf{n}' \times \mathbf{m}]$ and $\mathbf{m}'' = N[\mathbf{n}'' \times \mathbf{m}]$.
4. Return $\{\mathbf{m}, \mathbf{m}', \mathbf{m}''\}$,

In deriving intersection($\mathbf{Q}$, $\mathbf{n}$), we assumed that the conic $\mathbf{Q}$ was proper, but the conic $\mathbf{Q}$ of Eq. (31) becomes improper when $(\mathbf{n}', \mathbf{n}'') = 0$ or $\mathbf{n}' = \pm\mathbf{n}''$. However, the resulting 3D interpretation is correct as long as the computation does not fail. The only exception in which the computation fails is when $\mathbf{Qu} = \mathbf{0}$ in Step 2 of intersection($\mathbf{Q}$, $\mathbf{n}$). In this case, line l is contained in conic $\mathbf{Q}$, thereby yielding infinitely many solutions. This occurs, for example, when the projection of the three line forms a special type of "T-junction" (we omit the details).

Since a line generally intersects a conic at two points, there exist *two* solutions. This is intuitively evident because we do not know in which direction each line approaches the viewpoint. If we can tell this for one line segment, the 3D geometry of the three lines is uniquely determined.

EXAMPLE 1. Figure 5a is a real image (270 × 300 pixels) of a rectangular box. Figure 5b shows detected edges. The focal length is estimated to be $f = 1760$ (pixels) [11]. Applying the above procedure to the three edges forming a "Y-junction," we obtain the following two sets of 3D interpretations:

$$\mathbf{m} = \begin{pmatrix} 0.229 \\ -0.760 \\ 0.607 \end{pmatrix}, \quad \mathbf{m}' = \begin{pmatrix} 0.303 \\ 0.649 \\ 0.698 \end{pmatrix}, \quad \mathbf{m}'' = \begin{pmatrix} -0.925 \\ 0.025 \\ 0.379 \end{pmatrix}, \tag{36}$$

$$\mathbf{m} = \begin{pmatrix} -0.204 \\ 0.786 \\ 0.584 \end{pmatrix}, \quad \mathbf{m}' = \begin{pmatrix} -0.273 \\ -0.618 \\ 0.737 \end{pmatrix}, \quad \mathbf{m}'' = \begin{pmatrix} 0.940 \\ -0.009 \\ 0.341 \end{pmatrix}. \tag{37}$$

Here, the X-axis extends upward, the Y-axis rightward, and the Z-axis away from the viewer. The true solution is given by the first set. The second set gives the "mirror image" of the true solution with respect to a plane perpendicular to the N-vector of the junction vertex [6, 8]. The spurious solution can be removed if we apply the same procedure to other junctions and pick out an (almost) common solution.

The above procedure is very useful when the number of orthogonal edges is limited (say three), but if many orthogonal edges are available as in this example, the 3D interpretation is more robustly computed by estimating

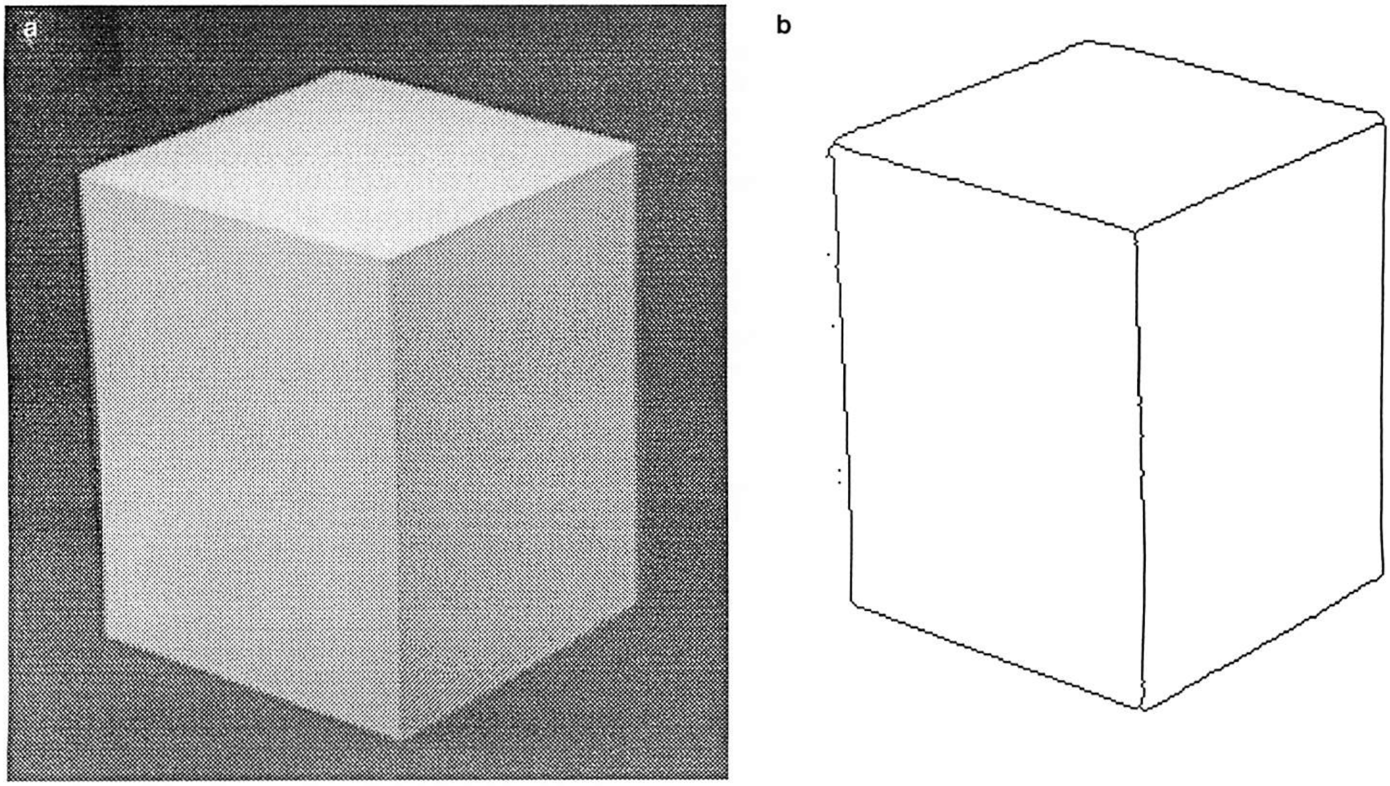

FIG. 5. (a) A real image of a rectangular box. (b) Detected edges.

vanishing points. Computing the vanishing points of the three sets of parallel edges as a common intersection by least-squares (we omit the details), we obtain the following 3D interpretation:

$$\mathbf{m} = \begin{pmatrix} 0.242 \\ -0.783 \\ 0.573 \end{pmatrix}, \quad \mathbf{m}' = \begin{pmatrix} 0.297 \\ 0.629 \\ 0.718 \end{pmatrix}, \quad \mathbf{m}'' = \begin{pmatrix} -0.920 \\ 0.029 \\ 0.391 \end{pmatrix}. \tag{38}$$

7. SUPPORTING PLANE AND THE TRUE SHAPE OF A CONIC

In the rest of this paper, we consider only proper conics. Consider a conic in the scene. We call the planar surface, on which the conic lies, the *supporting plane*. If the projection of the *center* (of symmetry) of the true shape is detected, the unit surface normal to the supporting plane is easily computed.

PROPOSITION 13. *If* $\mathbf{Q}$ *is a projection of a conic in the scene and if* $\mathbf{m}$ *is the N-vector of its projected center, the unit surface normal to its supporting plane is given by*

$$\mathbf{n} = \pm N[\mathbf{Qm}]. \tag{39}$$

Proof. Since the unit surface normal $\mathbf{n}$ to the supporting plane is the N-vector of its *vanishing line* [9], the assertion is immediately obtained if we can prove that the vanishing line l_∞ is the "polar" of the projected center Pc (Fig. 6a). This is proved as follows: Suppose the supporting plane is parallel to the image plane. Translate the supporting plane so that the center of the conic is on the Z-axis, and rotate it around the Z-axis so that the major and minor axes coincide with the image coordinate axes. The equation of the conic is then in canonical form, and the matrix $\mathbf{Q}$ representing it is diagonal. Since the N-vector of the projected center is $\mathbf{m} = (0, 0, \pm 1)^T$, the N-vector of its polar is $\pm N[\mathbf{Qm}] = (0, 0, \pm 1)^T$, which can be identified with the N-vector of the vanishing line at infinity. If the supporting plane is translated and rotated arbitrarily in the scene, the resulting transformation on the image plane is a "collineation" since (i) collinear points are mapped to collinear points, (ii) concurrent lines are mapped to concurrent lines, and (iii) incidence is preserved. Since the "polarity" is invariant to collineations (Propositions 7 and 8), the same relationship must hold for any projection of the conic. ■

COROLLARY 5. *If* $\mathbf{Q}$ *is a projection of a conic in the scene and if* $\mathbf{n}$ *is the unit surface normal to its supporting plane, the N-vector of the projected center is*

$$\mathbf{m} = \pm N[\mathbf{Q}^{-1}\mathbf{n}]. \tag{40}$$

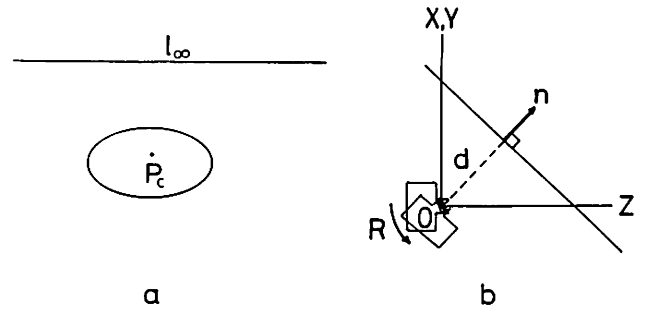

FIG. 6. (a) The polarity of a conic interpreted as the projected center and the vanishing line. (b) The camera is rotated so that the supporting plane becomes parallel to the image plane.

If the surface normal to the supporting plane is known, the true shape of the conic is easily computed from its projection. All we need to do is apply a "camera rotation transformation" defined in Section 2 such that the image plane is effectively parallel to the supporting plane (Fig. 6b).

THEOREM 3. *If* $\mathbf{n}$ *and* d *are the unit surface normal and the distance to the supporting plane of conic* $\mathbf{Q}$, *respectively, the true shape is given by*

$$\overline{\mathbf{Q}} = \mathbf{S}\mathbf{R}^T\mathbf{Q}\mathbf{R}\mathbf{S}, \tag{41}$$

where

$$\mathbf{R} = \begin{pmatrix} E & F & n_1 \\ F & G & n_2 \\ -n_1 & -n_2 & n_3 \end{pmatrix}, \quad \mathbf{S} = \left(\frac{f}{d}\right)^{1/3} \begin{pmatrix} 1 & & \\ & 1 & \\ & & d/f \end{pmatrix}, \tag{42}$$

$$E = \frac{n_1^2 n_3 + n_2^2}{n_1^2 + n_2^2}, \quad F = -\frac{n_1 n_2 (1 - n_3)}{n_1^2 + n_2^2}, \quad G = \frac{n_2^2 n_3 + n_1^2}{n_1^2 + n_2^2}. \tag{43}$$

Proof. Letting $\mathbf{k} = (0, 0, 1)^T$, define unit vector $\mathbf{l}$ and angle Ω by

$$\mathbf{l} = N[\mathbf{k} \times \mathbf{n}] = \begin{pmatrix} -n_2/\sqrt{n_1^2 + n_2^2} \\ n_1/\sqrt{n_1^2 + n_2^2} \\ 0 \end{pmatrix}, \quad \Omega = \cos^{-1} n_3. \tag{44}$$

If the camera is rotated around $\mathbf{l}$ by Ω, the Z-axis coincides with the direction of $\mathbf{n}$ (Fig. 6b). The matrix $\mathbf{R}$ representing this rotation is given by the first of Eqs. (42) [6, 8]. This camera rotation induces "collineation $\mathbf{R}$" on the image plane, which maps conic $\mathbf{Q}$ into conic $\mathbf{Q}' = \mathbf{R}^T\mathbf{Q}\mathbf{R}$ (Proposition 1). Now that the supporting plane is parallel to the image plane, conic $\mathbf{Q}'$ is similar to its true shape. Since the supporting plane is at distance d from the viewpoint O, the true shape $\overline{\mathbf{Q}}$ is obtained by expanding it by d/f. This transformation defines "collineation $\mathbf{S}$" as given in Eqs. (42) (the multiplier $(f/d)^{1/3}$ is due to our scaling convention that all collineations have determinants 1). If this transformation is added, we obtain Eq. (41). ∎

The true conic shape $\overline{\mathbf{Q}}$ is then transformed into its canonical form by an appropriate 2D Euclidean motion (Theorem 1).

EXAMPLE 2. Suppose we observe on the image plane a circle

$$x^2 + y^2 = r^2, \quad r > 0. \tag{45}$$

In matrix form,

$$\mathbf{Q} = \kappa \begin{pmatrix} 1 & & \\ & 1 & \\ & & -r^2/f^2 \end{pmatrix}, \tag{46}$$

where $\kappa = (f/r)^{2/3}$. Let

$$Z = kX + h, \quad h > 0, \tag{47}$$

be its supporting plane. The unit surface normal $\mathbf{n}$ and the distance d to it are

$$\mathbf{n} = \begin{pmatrix} -k/\sqrt{1+k^2} \\ 0 \\ 1/\sqrt{1+k^2} \end{pmatrix}, \quad d = \frac{h}{\sqrt{1+k^2}}. \tag{48}$$

The matrices $\mathbf{R}$ and $\mathbf{S}$ given by Eqs. (42) are

$$\mathbf{R} = \begin{pmatrix} 1/\sqrt{1+k^2} & 0 & -k/\sqrt{1+k^2} \\ 0 & 1 & 0 \\ k/\sqrt{1+k^2} & 0 & 1/\sqrt{1+k^2} \end{pmatrix}, \tag{49}$$

$$\mathbf{S} = \sigma \begin{pmatrix} 1 & & \\ & 1 & \\ & & h/f\sqrt{1+k^2} \end{pmatrix}, \tag{50}$$

where $\sigma = (f\sqrt{1+k^2}/h)^{1/3}$. The true shape $\overline{\mathbf{Q}}$ given by Eq. (41) has the form

$$\overline{\mathbf{Q}} = \bar{\kappa} \begin{pmatrix} (1 - k^2r^2/f^2)/(1+k^2) & 0 & -k(1 + r^2/f^2)h/f\sqrt{(1+k^2)^3} \\ 0 & 1 & 0 \\ -k(1 + r^2/f^2)h/f\sqrt{(1+k^2)^3} & 0 & (k^2 - r^2/f^2)h^2/f^2(1+k^2)^2 \end{pmatrix}, \tag{51}$$

where $\bar{\kappa} = (f^2\sqrt{1+k^2}/rh)^{2/3}$. Hence, the equation of this conic is

$$\frac{1 - k^2r^2/f^2}{1+k^2}x^2 + y^2 - \frac{2k(1 + r^2/f^2)}{\sqrt{(1+k^2)^3}}hx + \frac{k^2 - r^2/f^2}{(1+k^2)^2}h^2 = 0. \tag{52}$$

According to Theorem 1, we have

$$\lambda_1 = \frac{\bar{\kappa}(1 - k^2r^2/f^2)}{1+k^2}, \quad \lambda_2 = \bar{\kappa}, \quad \mu = \frac{h^2}{\bar{\kappa}f^2/r^2 - k^2}. \tag{53}$$

If $|k| < f/r$, $\overline{\mathbf{Q}}$ is an ellipse and reduces to the canonical form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad a = \frac{\sqrt{1+k^2}\,f/r}{f^2/r^2 - k^2}h, \quad b = \frac{h}{\sqrt{f^2/r^2 - k^2}}. \tag{54}$$

If $|k| > f/r$, $\overline{\mathbf{Q}}$ is a hyperbola and reduces to the canonical form

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \quad a = \frac{\sqrt{1+k^2}\,f/r}{k^2 - f^2/r^2}h, \quad b = \frac{h}{\sqrt{k^2 - f^2/r^2}}. \tag{55}$$

If $k = \pm f/r$, $\overline{\mathbf{Q}}$ is a parabola and reduces to the canonical form

$$y = \frac{f^2\sqrt{1 + r^2/f^2}}{2r^2h}x^2. \tag{56}$$

8. 3D INTERPRETATION OF A CIRCLE

Since many man-made objects have circular shapes, circles are important features for 3D object recognition. Also, the 3D orientation of a circle in the scene can be computed from its projection, and an analytical procedure was given by Forsyth *et al.* [2]. Here, we reiterate it as preparation for the case involving ellipses. Recall that a proper conic $\mathbf{Q}$ has signature (2, 1) (Proposition 5): it has two positive eigenvalues, λ_1 and λ_2, and one negative eigenvalue, λ_3.

THEOREM 4. *If conic* $\mathbf{Q}$ *is a projection of a circle of* *radius r, let* λ_1, λ_2, *and* λ_3 *be the eigenvalues of* $\mathbf{Q}$ ($\lambda_3 < 0 < \lambda_1 \le \lambda_2$), *and* $\mathbf{u}_2$ *and* $\mathbf{u}_3$ *the unit eigenvectors for eigenvalues* λ_2 *and* λ_3, *respectively. The unit surface normal* $\mathbf{n}$ *to the supporting plane is given by*

$$\mathbf{n} = \sqrt{(\lambda_2 - \lambda_1)/(\lambda_2 - \lambda_3)}\,\mathbf{u}_2 + \sqrt{(\lambda_1 - \lambda_3)/(\lambda_2 - \lambda_3)}\,\mathbf{u}_3. \tag{57}$$

Its distance is

$$d = \lambda_1^{3/2} r. \tag{58}$$

Proof. Suppose conic $\mathbf{Q}$ is in the canonical form

$$x^2 + \alpha y^2 = \lambda, \quad \alpha \ge 1, \quad \lambda > 0. \tag{59}$$

In matrix form,

$$\mathbf{Q} = \kappa \begin{pmatrix} 1 & & \\ & \alpha & \\ & & -\gamma/f^2 \end{pmatrix}, \tag{60}$$

where $\kappa = (f/\sqrt{\alpha\gamma})^{2/3}$. Since $\alpha \ge 1$, the major axis is along the X-axis, and the circle is slanted in the direction of the minor axis. Hence, its surface normal $\mathbf{n}$ has the form $\mathbf{n} = (0, \sin\theta, \cos\theta)^{\mathrm{T}}$. Due to the symmetry with respect to the x-axis, if θ gives a solution, so does $-\theta$. So, assume that $0 \le \theta \le \pi/2$. Now, we apply the "camera rotation transformation"; we rotate the camera (or the coordinate system) around the X-axis by angle $-\theta$ so that the supporting plane becomes parallel to the image plane (Fig. 6). The corresponding rotation matrix is

$$\mathbf{R} = \begin{pmatrix} 1 & & \\ & \cos\theta & \sin\theta \\ & -\sin\theta & \cos\theta \end{pmatrix}. \tag{61}$$

After this transformation, we observe on the image plane a circle in the form

$$x^2 + (y + c)^2 = \rho^2, \quad c \ge 0, \rho > 0. \tag{62}$$

In matrix form,

$$\mathbf{Q}' = \kappa' \begin{pmatrix} 1 & & \\ & 1 & c/f \\ & c/f & (c^2 - \rho^2)/f^2 \end{pmatrix}, \tag{63}$$

where $\kappa' = (f/\rho)^{2/3}$. The camera rotation by $\mathbf{R}$ induces "collineation $\mathbf{R}$" on the image plane, which maps conic $\mathbf{Q}$ to conic $\mathbf{Q}' = \mathbf{R}^{\mathrm{T}}\mathbf{Q}\mathbf{R}$ (Proposition 1). Hence, conic $\mathbf{Q}'$ also has the form

$$\mathbf{Q}' = \kappa \begin{pmatrix} 1 & & \\ & \cos\theta & -\sin\theta \\ & \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 1 & & \\ & \alpha & \\ & & -\gamma/f^2 \end{pmatrix} \begin{pmatrix} 1 & & \\ & \cos\theta & \sin\theta \\ & -\sin\theta & \cos\theta \end{pmatrix}. \tag{64}$$

Comparing the (1, 1) elements, we immediately find that

$$\rho = \sqrt{\alpha\gamma}. \tag{65}$$

The remaining submatrices satisfy

$$\begin{pmatrix} \alpha & \\ & -\gamma/f^2 \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 1 & c/f \\ c/f & (c^2 - \rho^2)/f^2 \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}. \tag{66}$$

This means that matrix

$$\begin{pmatrix} 1 & \\ c/f & (c^2 - \rho^2)/f^2 \end{pmatrix} \tag{67}$$

has eigenvalues α (≥ 1) and $-\gamma/f^2$ (<0) with corresponding eigenvectors $(\cos\theta, \sin\theta)^{\mathrm{T}}$ and $(-\sin\theta, \cos\theta)^{\mathrm{T}}$, respectively. Since the trace and the determinant are invariant, we have

$$\alpha - \frac{\gamma}{f^2} = 1 + \frac{c^2 - \rho^2}{f^2}, \quad -\frac{\alpha\gamma}{f^2} = \frac{c^2 - \rho^2}{f^2} - \frac{c^2}{f^2}, \tag{68}$$

from which we again obtain Eq. (65) and

$$c = \sqrt{(\alpha - 1)(\gamma + f^2)}. \tag{69}$$

Since θ is the orientation of the eigenvector for eigenvalue α, we have

$$\tan\theta = \frac{\alpha - 1}{c/f} = \sqrt{(\alpha - 1)/(1 + \gamma/f^2)}. \tag{70}$$

Since $0 \le \theta \le \pi/2$ we obtain

$$\sin\theta = \sqrt{(\alpha - 1)/(\alpha + \gamma/f^2)},$$
$$\cos\theta = \sqrt{(1 + \gamma/f^2)/(\alpha + \gamma/f^2)}. \tag{71}$$

The distance d is given by

$$d = \frac{fr}{\rho} = \frac{fr}{\sqrt{\alpha\gamma}}. \quad (72)$$

If the original conic $\mathbf{Q}$ is not in the canonical form of Eq. (59), we can find an orthogonal matrix $\mathbf{U}$ such that

$$\mathbf{U}^{\mathrm{T}}\mathbf{Q}\mathbf{U} = \begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{pmatrix},$$

$$\lambda_3 < 0 < \lambda_1 \leq \lambda_2, \ \lambda_1\lambda_2\lambda_2 = -1. \quad (73)$$

This can be interpreted as a (generalized) "camera rotation transformation" defined by a camera rotation (and a reflection) by $\mathbf{U}$. If the camera (or the coordinate system) is rotated (and reflected) around the viewpoint O by $\mathbf{U}$, the conic on the image plane becomes

$$x^2 + \frac{\lambda_2}{\lambda_1} y^2 = -f^2 \frac{\lambda_3}{\lambda_1}. \quad (74)$$

Hence, if we define

$$\alpha = \frac{\lambda_2}{\lambda_1} (\geq 1), \quad \lambda = -f^2 \frac{\lambda_3}{\lambda_1} (>0), \quad (75)$$

Eq. (74) has the form of Eq. (59). The unit vectors along the Y- and Z-axes in this position are the second and third columns of matrix $\mathbf{U}$, which are the eigenvectors $\mathbf{u}_2$ and $\mathbf{u}_3$ of $\mathbf{Q}$ for eigenvalues λ_2 and λ_3, respectively. Hence, the unit surface normal to the supporting plane is given by

$$\mathbf{n} = \mathbf{u}_2 \sin\theta + \mathbf{u}_3 \cos\theta. \quad (76)$$

Substituting Eqs. (75) into Eqs. (71) and (72), we obtain Eqs. (57) and (58). ∎

Note that the signs of the eigenvectors $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$ are arbitrary. Since $\mathbf{n}$ and $-\mathbf{n}$ indicate the same surface orientation, the number of 3D interpretations is as follows:

1. If $\lambda_1 \neq \lambda_2$, *two* interpretations exist.
2. If $\lambda_1 = \lambda_2$, only *one* interpretation exists.

EXAMPLE 3. Suppose we are viewing an ellipse $\mathbf{Q}$ given by

$$4x^2 + 16y^2 = 1. \quad (77)$$

In matrix form,

$$\mathbf{Q} = \kappa \begin{pmatrix} 4 & & \\ & 16 & \\ & & -1/f^2 \end{pmatrix}, \quad (78)$$

where $\kappa = (f/8)^{2/3}$. This is in canonical form. Hence, $\mathbf{u}_1 = (\pm 1, 0, 0)^{\mathrm{T}}$, $\mathbf{u}_2 = (0, \pm 1, 0)^{\mathrm{T}}$, and $\mathbf{u}_3 = (0, 0, \pm 1)^{\mathrm{T}}$. The corresponding eigenvalues are $\lambda_1 = 4\kappa$, $\lambda_2 = 16\kappa$, and $\lambda_3 = -\kappa/f^2$. If this ellipse is a projection of a circle of radius r, the unit normal to the supporting plane and the distance to it are determined from Eqs. (57) and (58) in the form

$$\mathbf{n} = \frac{1}{\sqrt{16 + 1/f^2}} \begin{pmatrix} 0 \\ \pm 2\sqrt{3} \\ \pm\sqrt{4 + 1/f^2} \end{pmatrix}, \quad d = fr, \quad (79)$$

where the two double signs are independent. This means that the circle is slanted relative to the image plane in the y-direction by angle

$$\theta = \sin^{-1} \sqrt{12/(16 + 1/f^2)}. \quad (80)$$

According to Corollary 5, the N-vector of the projected center is

$$\mathbf{m} = \pm N[\mathbf{Q}^{-1}\mathbf{n}] = N\left[\begin{pmatrix} 0 \\ \pm\sqrt{3}/8 \\ \pm f^2\sqrt{4 + 1/f^2} \end{pmatrix}\right]. \quad (81)$$

The corresponding image coordinates are

$$\left(0, \frac{\pm\sqrt{3}}{8f\sqrt{1 + 4f^2}}\right). \quad (82)$$

In the limit as $f \to \infty$, the slant angle approaches $\theta = \pi/3$ and the projected center approaches the image origin.

EXAMPLE 4. Figure 7a is a real image (512 × 512 pixels) of three coplanar circles of known radii. The focal length is estimated to be $f = 630$ (pixels) [11]. The supporting plane is manually placed so that supposedly $\mathbf{n} = (0.000, 0.707, 0.707)^{\mathrm{T}}$ and $d = 25$ (cm). Since the three circles share a common supporting plane, Theorem 4 should ideally predict an identical supporting plane for all the three circles, but this is not expected due to inaccuracy of camera calibration and image processing. Table 1 shows the computed unit surface normal $\mathbf{n}$ and the distance d (cm) to the supporting plane of each circle. The pairwise discrepancies in the orientations corresponding to the true solutions are less than 2.2°, while the pairwise discrepancies in the distances are less than 3%. Figure 7b

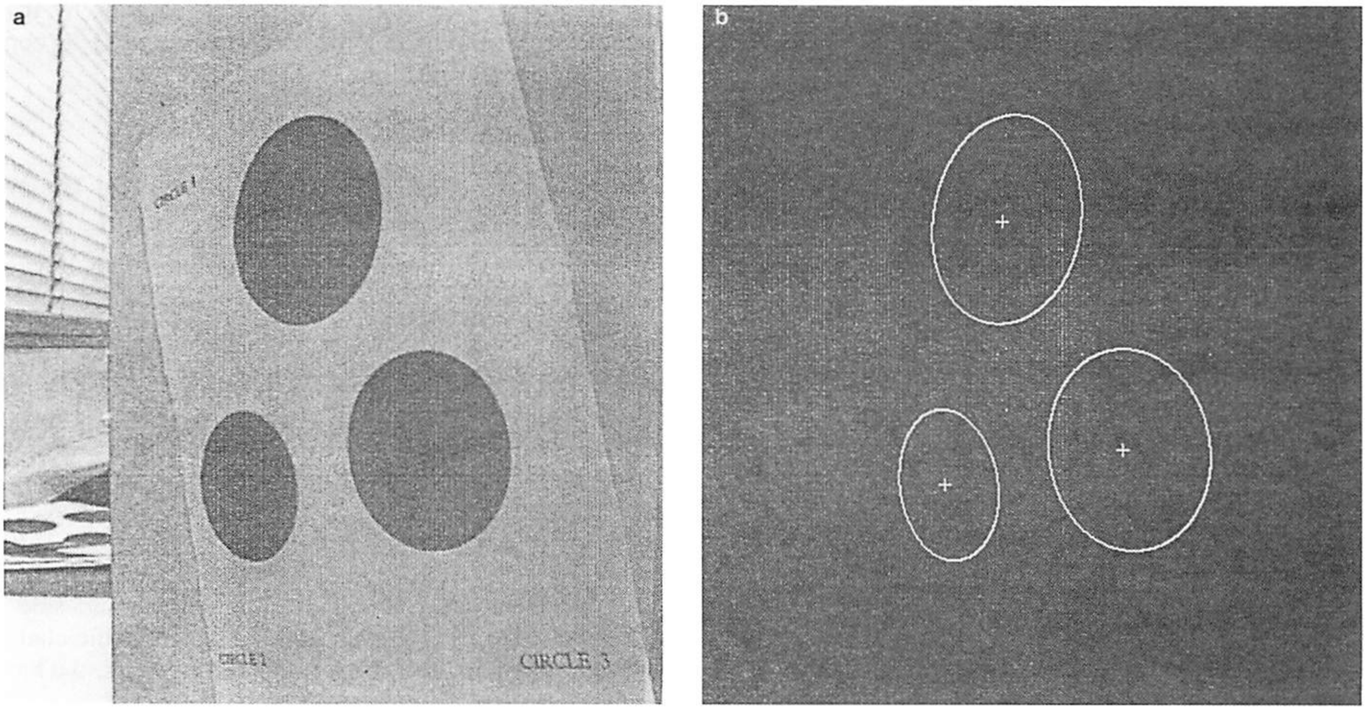

FIG. 7. (a) A real image of three coplanar circles. (b) Estimated projected centers.

shows the estimated projected centers computed by Corollary 5.

9. 3D INTERPRETATION OF AN ELLIPSE

Suppose the conic **Q** we observe on the image plane is known to be a projection of an ellipse of a known size and shape. In this case, its 3D geometry cannot be reconstructed uniquely: it is reconstructed only up to one free parameter. If the free parameter is identified with the distance d to the supporting plane, the unit surface normal **n** to the supporting plane is computed as follows:

THEOREM 5. *If conic* **Q** *is a projection of an ellipse of eccentricity* e $(0 < e < 1)$ *and area* S, *and if* d *is the distance to its supporting plane, the unit surface normal* **n** *to the supporting plane is computed by the following procedure:*

1. *Let* λ_1, λ_2, *and* λ_3 *be the eigenvalues of* **Q** $(\lambda_3 < 0 < \lambda_1 \le \lambda_2)$, *and* $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ *the orthonormal set of corresponding eigenvectors. Let*

$$\bar{e} = \frac{1}{1 - e^2} \ (>1). \tag{83}$$

2. *If*

$$\sqrt{\frac{S}{\pi}} \frac{\lambda_1^{3/2}}{\bar{e}^{3/4}} \le d \le \sqrt{\frac{S}{\pi}} \min\left(\bar{e}^{3/4}\lambda_1^{3/2}, \frac{\lambda_2^{3/2}}{\bar{e}^{3/4}}\right) \tag{84}$$

is not satisfied, conic **Q** *cannot be interpreted as a perspective projection of an ellipse of eccentricity* e *and area* S. *If it is satisfied,*

3. *Define* k *and function* $\psi(x)$ *by*

$$k = \frac{1}{\bar{e}^{1/2}\lambda_1}\left(\frac{\pi d^2}{S}\right)^{1/3},$$

$$\psi(x) = (x - 1)\left(x - \frac{\lambda_2}{\lambda_1}\right)\left(x - \frac{\lambda_3}{\lambda_1}\right). \tag{85}$$

4. *Define function* $\mu(x)$ *by*

$$\mu(x) = \frac{1}{\sqrt{A/(x-1)^2 + B/(x-\bar{e})^2 + 1}}, \quad x \ne 1, \bar{e}, \tag{86}$$

$$A = \frac{\psi(k)}{(\bar{e} - 1)k^3}, \quad B = -\frac{\psi(\bar{e}k)}{(\bar{e} - 1)k^3}, \tag{87}$$

TABLE 1
Computed Unit Surface Normal n and Distance d (cm) to the Supporting Plane

	n		d
1	(−0.004, 0.695, 0.719)	(0.218, −0.693, 0.687)	24.93
2	(0.012, 0.710, 0.704)	(−0.201, −0.794, 0.574)	25.67
3	(−0.021, 0.696, 0.718)	(−0.419, −0.476, 0.867)	25.44

and extend the domain of $\mu(x)$ *by defining* $\mu(1) = \mu(\bar{e}) = 0$.

5. *If* $\lambda_1 \neq \lambda_2$, *the unit surface normal* $\mathbf{n}$ *to the supporting plane is given by*

$$\mathbf{n} = \mu\left(\frac{1}{k}\right)\mathbf{u}_1 + \mu\left(\frac{\lambda_2}{k\lambda_1}\right)\mathbf{n}_2 + \mu\left(\frac{\lambda_3}{k\lambda_1}\right)\mathbf{u}_3, \quad (88)$$

If $\lambda_1 = \lambda_2$,

$$\mathbf{n} = \sqrt{1 - \mu(\lambda_3/k\lambda_1)^2}\mathbf{u}_1 + \mu\left(\frac{\lambda_3}{k\lambda_1}\right)\mathbf{u}_3. \quad (89)$$

Proof. Suppose the conic $\mathbf{Q}$ is in the canonical form

$$x^2 + \alpha y^2 = \gamma, \quad \alpha \geq 1, \gamma > 0. \quad (90)$$

In matrix form,

$$\mathbf{Q} = \kappa \begin{pmatrix} 1 & & \\ & \alpha & \\ & & -\gamma/f^2 \end{pmatrix}, \quad (91)$$

where $\kappa = (f/\sqrt{\alpha\gamma})^{2/3}$. Let $\mathbf{n} = (n_1, n_2, n_3)^T$ be the surface normal to the supporting plane.

As in the case of a circle, we apply the "camera rotation transformation": we rotate the camera (or the coordinate system) by $\mathbf{R}$ so that the optical axis is aligned with the surface normal $\mathbf{n}$ (Fig. 6). This means that the third column of $\mathbf{R}$ is $\mathbf{n}$. In this new camera position, the observed conic $\mathbf{Q}'$ has a shape similar to the true shape: it has eccentricity e and area f^2S/d^2. Let (a, b) be the center of the conic $\mathbf{Q}'$. The conic has the form

$$(x - a)^2 + \bar{e}(y - b)^2 = \bar{e}^{1/2}f^2S/\pi d^2, \quad (92)$$

or

$$x^2 + \bar{e}y^2 - 2f(\bar{a}x + \bar{e}\bar{b}y) + f^2c = 0, \quad (93)$$

where

$$\bar{a} = \frac{a}{f}, \quad \bar{b} = \frac{b}{f}, \quad c = \frac{a^2}{f^2} + \bar{e}\frac{b^2}{f^2} - \bar{e}^{1/2}\frac{S}{\pi d^2}. \quad (94)$$

In matrix form,

$$\mathbf{Q}' = \kappa' \begin{pmatrix} 1 & & -\bar{a} \\ & \bar{e} & -\bar{e}\bar{b} \\ -\bar{a} & -\bar{e}\bar{b} & c \end{pmatrix}, \quad (95)$$

where $\kappa' = (\pi d^2/S)^{1/3}/\bar{e}^{1/2}$. The camera rotation by $\mathbf{R}$ induces "collineation $\mathbf{R}$," which maps conic $\mathbf{Q}$ to conic $\mathbf{Q}' = \mathbf{R}^T\mathbf{Q}\mathbf{R}$ (Proposition 1). Hence,

$$\begin{pmatrix} 1 & & -\bar{a} \\ & \bar{e} & -\bar{e}\bar{b} \\ -\bar{a} & -\bar{e}\bar{b} & c \end{pmatrix} = \frac{1}{k}\mathbf{R}^T \begin{pmatrix} 1 & & \\ & \alpha & \\ & & -\gamma/f^2 \end{pmatrix} \mathbf{R}, \quad (96)$$

where

$$k = \frac{1}{\bar{e}^{1/2}}\left(\frac{\pi d^2}{S}\right)^{1/3}\left(\frac{\sqrt{\alpha\gamma}}{f}\right)^{2/3}. \quad (97)$$

If we define a cubic polynomial $\psi(\lambda)$ by

$$\psi(\lambda) = (\lambda - 1)(\lambda - \alpha)(\lambda + \gamma/f^2), \quad (98)$$

the characteristic polynomial of the right-hand side of Eq. (96) is $\psi(k\lambda)/k^3$, which must be equal to the characteristic polynomial of the left-hand side. Thus, we have

$$\frac{1}{k^3}\psi(k\lambda) = \begin{vmatrix} \lambda - 1 & & \bar{a} \\ & \lambda - \bar{e} & \bar{e}\bar{b} \\ \bar{a} & \bar{e}\bar{b} & \lambda - c \end{vmatrix}. \quad (99)$$

Substituting $\lambda = 1$ and $\lambda = \bar{e}$, we obtain

$$\frac{\psi(k)}{k^3} = -(1 - \bar{e})\bar{a}^2, \quad \frac{\psi(k\bar{e})}{k^3} = -(\bar{e} - 1)\bar{e}^2\bar{b}^2. \quad (100)$$

Since $\bar{e} > 1$, such an $\bar{a}$ and $\bar{b}$ exist if and only if

$$\psi(k) \geq 0, \quad \psi(\bar{e}k) \leq 0. \quad (101)$$

From Eq. (98), the first condition is equivalent to $0 < k \leq 1$ or $k \geq \alpha$, and the second one is equivalent to $1/\bar{e} \leq k \leq \alpha/\bar{e}$. Since $\bar{e} \geq 1$, this means that

$$\frac{1}{\bar{e}} \leq k \leq \min\left(1, \frac{\alpha}{\bar{e}}\right). \quad (102)$$

If this condition is satisfied, we obtain

$$\bar{a}_2 = \frac{\psi(k)}{(\bar{e} - 1)k^3}(\geq 0), \quad \bar{e}^2\bar{b}^2 = \frac{-\psi(\bar{e}k)}{(\bar{e} - 1)k^3}(\geq 0). \quad (103)$$

Assume that $\alpha \neq 1$. Then, the three eigenvalues $1/k$, α/k, and $-\gamma/f^2k$ of the matrix of Eq. (96) are all simple roots. The unit eigenvector $\mathbf{u} = (u_1, u_2, u_3)^T$ for eigenvalue $1/k$ is determined by

$$\begin{pmatrix} 1/k - 1 & & \bar{a} \\ & 1/k - \bar{e} & \bar{e}\bar{b} \\ \bar{a} & \bar{e}\bar{b} & 1/k - c \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \quad (104)$$

Since $\bar{e} \neq 1$, the first and second rows of the matrix are linearly independent unless $k = 1, 1/\bar{e}$. If $k = 1$, then $\psi(k) = 0$ from Eq (98), so $\bar{a} = 0$ due to Eqs. (103). Hence, the first row is identically 0. However, since $1/k$ is a simple root, the second and third rows must be linearly independent. It follows that the normalized eigenvector is $\mathbf{u} = (\pm 1, 0, 0)^T$. If $k = 1/\bar{e}$, we obtain $\mathbf{u} = (0, \pm 1, 0)^T$ by the same argument. In the remaining cases $k \neq 1, 1/\bar{e}$, the first and second rows are linearly independent, so we obtain

$$\mathbf{u} = \pm N\left[\begin{pmatrix} -\bar{a}/(1/k - 1) \\ -\bar{e}\bar{b}(1/k - \bar{e}) \\ 1 \end{pmatrix}\right]. \quad (105)$$

Hence, if we define function

$$\mu(x) = \frac{1}{\sqrt{\bar{a}^2/(x-1)^2 + \bar{e}^2\bar{b}^2/(x-\bar{e})^2 + 1}}, \quad x \neq 1, \bar{e}, \quad (106)$$

the third component u_3 is given by

$$u_3 = \pm\mu(1/k). \quad (107)$$

The cases $k = 1, 1/\bar{e}$ can be included in Eq. (107) if we define $\mu(1) = \mu(\bar{e}) = 0$.

The unit eigenvector $\mathbf{v} = (v_1, v_2, v_3)^T$ for eigenvalue α/k is determined by

$$\begin{pmatrix} \alpha/k - 1 & & \bar{a} \\ & \alpha/k - \bar{e} & \bar{e}\bar{b} \\ \bar{a} & \bar{e}\bar{b} & \alpha/k - c \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \quad (108)$$

Again, the first and second rows are linearly independent unless $k = \alpha, \alpha/\bar{e}$. In both cases, we obtain $v_3 = 0$ by the same argument shown earlier. Otherwise, we obtain

$$v_3 = \pm\mu(\alpha/k). \quad (109)$$

The exceptional cases are also included by the extension $\mu(1) = \mu(\bar{e}) = 0$.

The unit eigenvector $\mathbf{w} = (w_1, w_2, w_3)^T$ for eigenvalue $-\gamma/f^2k$ is determined by

$$\begin{pmatrix} -\gamma/f^2k - 1 & & \bar{a} \\ & -\gamma/f^2k - e & \bar{e}\bar{b} \\ \bar{a} & \bar{e}\bar{b} & -\gamma/f^2k - c \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \quad (110)$$

Since the first and second rows are always linearly independent, we obtain

$$w_3 = \pm\mu(-\gamma/f^2k). \quad (111)$$

From Eq. (96), we have $\mathbf{R}^T = (\mathbf{u}, \mathbf{v}, \mathbf{w})$ (the matrix having $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ as its columns in that order). Since the unit surface normal $\mathbf{n}$ is given by the third column of $\mathbf{R}$, we obtain $\mathbf{n} = (u_3, v_3, w_3)^T$.

Consider the remaining case $\alpha = 1$. In this case, Eq. (90) is a circle of radius $\sqrt{\gamma}$ centered at the image origin. Due to the circular symmetry, the X- and Y-components of $\mathbf{n} = (n_1, n_2, n_3)^T$ are indeterminate. Hence, we obtain

$$\mathbf{n} = \pm\begin{pmatrix} \sqrt{1 - n_3^2}\cos\phi \\ \sqrt{1 - n_3^2}\sin\phi \\ n_3 \end{pmatrix}, \quad 0 \leq \phi < 2\pi. \quad (112)$$

The third component n_3 is given by w_3 of Eq. (111).

If the original conic $\mathbf{Q}$ is not in the canonical form of Eq. (90), we can find an orthogonal matrix $\mathbf{U}$ such that

$$\mathbf{U}^T\mathbf{Q}\mathbf{U} = \begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{pmatrix},$$
$$\lambda_3 < 0 < \lambda_1 \leq \lambda_2, \; \lambda_1\lambda_2\lambda_3 = -1. \quad (113)$$

This can be interpreted as a (generalized) "camera rotation transformation" defined by $\mathbf{U}$. If the camera (or the coordinate system) is rotated (and reflected) around the viewpoint O by $\mathbf{U}$, the conic on the image plane becomes

$$x^2 + \frac{\lambda_2}{\lambda_1}y^2 = -f^2\frac{\lambda_3}{\lambda_1}. \quad (114)$$

Thus, if we define

$$\alpha = \frac{\lambda_2}{\lambda_1}\,(\geq 1), \quad \gamma = -f^2\frac{\lambda_3}{\lambda_1}\,(>0), \quad (115)$$

Eq. (114) has the form of Eq. (90). Hence, all we need to do is replace α and γ in all equations by Eqs. (115). The unit vectors along the X-, Y- and Z-axes in this position are the three columns of matrix $\mathbf{U}$, i.e., the unit eigenvectors $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$ of $\mathbf{Q}$ for eigenvalues λ_1, λ_2, and λ_3, respectively. The case $\alpha = 1$ corresponds to the degeneracy $\lambda_1 = \lambda_2$, in which case the eigenvectors $\mathbf{u}_1$ and $\mathbf{u}_2$ are arbitrary as long as they are orthogonal to each other and to $\mathbf{u}_3$. Hence, we do not lose generality if we set $\phi = 0$ in Eq. (112). ∎

Since the signs of the eigenvectors $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$ are arbitrary, Eq. (88) gives *eight* solutions. However, degeneracy occur if $\mu(1/k) = 0$ and/or $\mu(\lambda_2/k\lambda_1) = 0$. If $\mathbf{n}$ and $-\mathbf{n}$ are regarded as indicating the same surface orientation, the number of 3D interpretations for each d is as follows:

1. If $\lambda_1 \neq \lambda_2$,
   (a) If $k \neq 1$, $1/\bar{e}$, $\lambda_2/\bar{e}\lambda_1$, *four* interpretations exist,
   (b) if $k = 1$, $1/\bar{e}$, $\lambda_2/\bar{e}\lambda_1$,
      i. if $\lambda_2/\lambda_1 \neq \bar{e}$, *two* interpretations exist,
      ii. if $\lambda_2/\lambda_1 = \bar{e}$, only *one* interpretation exists.
2. If $\lambda_1 = \lambda_2$, one family of *infinitely many* axially symmetric interpretations exist.

Although the solution is not unique, this theorem has many potential applications. For example, the inequality (84) determines the possible range of the distance to the supporting plane. If two projections of coplanar ellipses of known shapes are observed, the distance to the common supporting plane can be obtained by searching for the value of d that yields the same (or in practice the closest) surface normals for both ellipses.

EXAMPLE 5. Suppose we observe a circle of radius r centered at the image origin:

$$x^2 + y^2 = r^2. \tag{116}$$

In matrix form,

$$\mathbf{Q} = \kappa \begin{pmatrix} 1 & & \\ & 1 & \\ & & -r^2/f^2 \end{pmatrix}, \tag{117}$$

where $\kappa = (f/r)^{2/3}$. Suppose this circle is a projection of an ellipse of eccentricity e and area S. Equation (116) is itself in canonical form. Hence, we can take $\mathbf{u}_1 = (\pm 1, 0, 0)^T$, $\mathbf{u}_2 = (0, \pm 1, 0)^T$, and $\mathbf{u}_3 = (0, 0, \pm 1)^T$. The corresponding eigenvalues are $\lambda_1 = \lambda_2 = \kappa$ and $\lambda_3 = -\kappa r^2/f^2$. From Eq. (84), the distance d to the supporting plane is *uniquely* determined:

$$d = \frac{f}{\bar{e}^{3/4}} \sqrt{S/\pi r^2}. \tag{118}$$

Since $\lambda_1 = \lambda_2$, the surface normal $\mathbf{n}$ to the supporting plane is determined from Eq. (89) in the form

$$\mathbf{n} = \frac{\pm 1}{\sqrt{1 + r^2/f^2}} \begin{pmatrix} e \\ 0 \\ \sqrt{1 - e^2 + r^2/f^2} \end{pmatrix}, \tag{119}$$

to which an arbitrary rotation around the Z-axis can be added. Hence, the supporting plane is slanted relative to the image plane by angle

$$\theta = \sin^{-1} \frac{e}{\sqrt{1 + r^2/f^2}}. \tag{120}$$

Its tilt orientation is indeterminate. According to Corollary 5, the N-vector of the projected center is

$$\mathbf{m} = \pm N[\mathbf{Q}^{-1}\mathbf{n}] = \pm N \left[\begin{pmatrix} er/f \\ 0 \\ -\sqrt{1 + (1 - e^2)f^2/r^2} \end{pmatrix} \right], \tag{121}$$

to which an arbitrary rotation around the Z-axis can be added. This means that the projected center is at distance

$$\frac{er}{\sqrt{1 + (1 - e^2)f^2/r^2}} \tag{122}$$

from the image origin. In the limit as $e \to 0$, the surface normal $\mathbf{n}$ approaches $(0, 0, \pm 1)$. In the limit as $f \to \infty$, the slant angle approaches $\theta = \sin^{-1} e$, and the projected center approaches the image origin.

EXAMPLE 6. Figure 8a is a real image (512 × 512 pixels) of three coplanar ellipses of known shapes. The common supporting plane is the same as that in Fig. 7a. If Theorem 5 is applied, inequality (84) restricts the distance d (cm) to the interval

$$16.59 < d < 26.41. \tag{123}$$

TABLE 2
Computed Unit Surface Normal n to the Supporting Plane

	n	
1	(−0.001, 0.701, 0.713)	(−0.044, 0.479, 0.759)
	(0.622, −0.569, 0.538)	(0.182, −0.791, 0.583)
2	(−0.012, 0.702, 0.712)	(0.120, 0.670, 0.733)
	(−0.218, −0.761, 0.611)	(−0.349, −0.728, 0.590)
3	(0.018, 0.693, 0.721)	(0.286, 0.611, 0.738)
	(−0.053, −0.472, 0.880)	(−0.321, −0.391, 0.863)

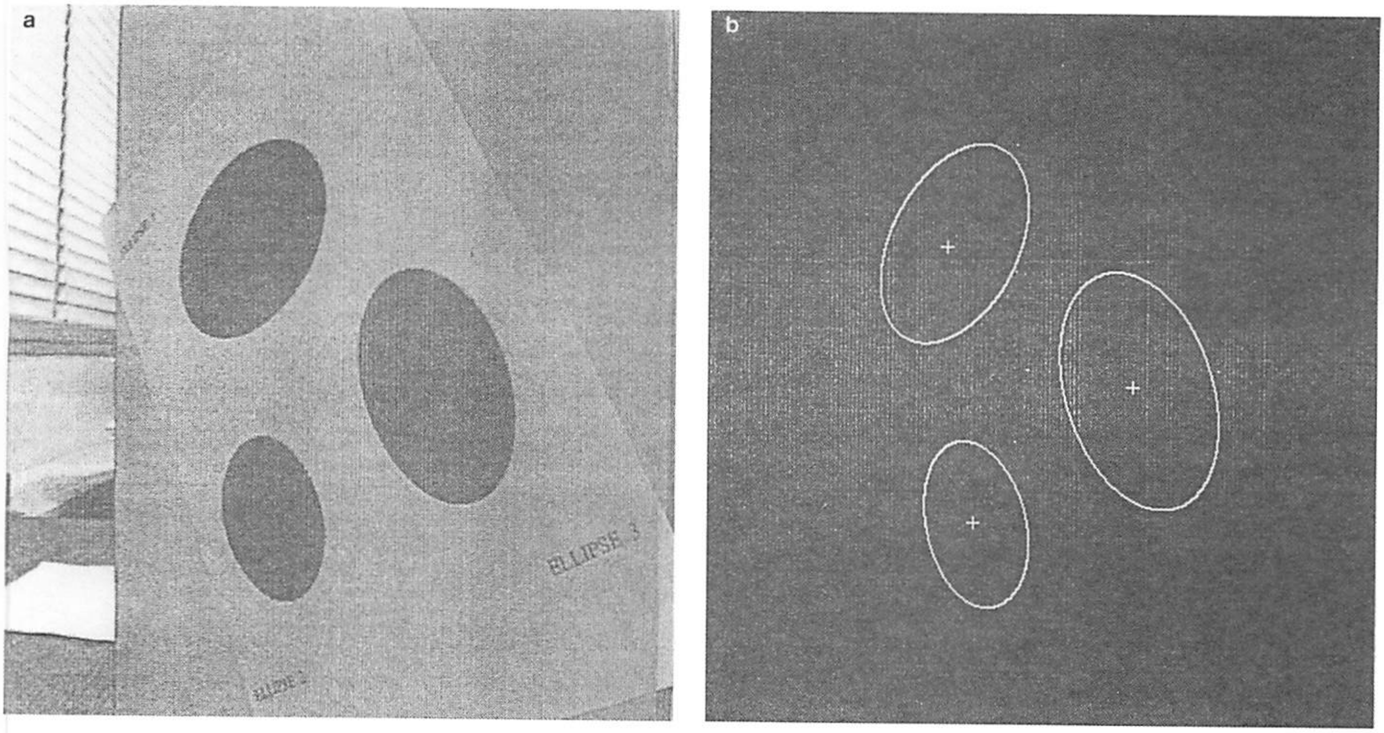

FIG. 8. (a) A real image of three coplanar ellipses. (b) Estimated projected centers.

If the average of the distances in Table 1 is used, the unit surface normal **n** to the supporting plane for each ellipse is computed as shown in Table 2. The pairwise discrepancies in the orientations corresponding to the true solution are less than 1.9°.

10. CONCLUDING REMARKS

We have presented a theory of computation involving conics by following the formulation of "computational projective geometry" presented by Kanatani [9]. First, we showed that the problem of interpreting the 3D geometry of three orthogonal lines can be succinctly described in the framework of computational projective geometry involving conics. We also derived computational procedures to interpret the 3D geometry of conics in the scene from their projections. Real image examples were also given to observe the accuracy of the computation.

ACKNOWLEDGMENTS

The first author thanks Michael Brady, Andrew Blake, and Andrew Zisserman of the University of Oxford, Steven Maybank of the GEC Hirst Research Center (U.K.), and Azriel Rosenfeld, Larry Davis, Yiannis Aloimonos, and Rama Chellappa of the University of Maryland for various discussions.

REFERENCES

1. S. T. Barnard, Choosing a basis for perceptual space, *Comput. Vision Graphics Image Process.* **29** (1985), 1233–1251.
2. D. Forsyth, J. L. Mundy, A. Zisserman, C. Coelho, A. Heller, and C. Rothwell, Invariant descriptors for 3-D object recognition and pose, *IEEE Trans. Pattern Anal. Mach. Intell.* **13** (1991), 971–991.
3. R. Horaud, New methods for matching 3-D objects with single perspective views, *IEEE Trans. Pattern Anal. Mach. Intell.* **9** (1987), 401–412.
4. T. Kanade, Recovery of three-dimensional shape of an object from a single view, *Artif. Intell.* **17** (1981), 409–461.
5. K. Kanatani, Camera rotation invariance of image characteristics, *Comput. Vision Graphics Image Process.* **39** (1987), 328–354.
6. K. Kanatani, Constraints on length and angle, *Comput. Vision Graphics Image Process.* **41** (1988), 28–42.
7. K. Kanatani, Transformation of optical flow by camera rotation, *IEEE Trans. Pattern Anal. Mach. Intell.* **10** (1988), 131–143.
8. K. Kanatani, *Group-Theoretical Methods in Image Understanding*, Springer-Verlag, Berlin, Germany, 1990.
9. K. Kanatani, Computational projective geometry, *CVGIP: Image Understanding* **54** (1991), 333–348.
10. K. Kanatani, *Geometric Computation for Machine Vision*, Oxford Univ. Press, Oxford, 1993.
11. K. Kanatani and Y. Onodera, Anatomy of camera calibration using vanishing points, *IEICE Trans. Inform. System* **74**, No. 10 (1991), 3369–3378.
12. J. G. Semple and G. T. Kneebone, *Algebraic Projective Geometry*, Clarendon Press, Oxford, 1952 (reprinted 1979).
13. T. Shakunaga and H. Kaneko, Perspective angle transform: Principle of shape from angle, *Int. J. Comput. Vision* **3** (1989), 239–254.