

MEASUREMENT OF CRACK DISTRIBUTION IN A ROCK MASS FROM OBSERVATION OF ITS SURFACES

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ABSTRACT

The anisotropy of the crack distribution in a rock mass is characterized by what is termed the "fabric tensor," and its geometrical interpretation is given. A practical procedure is presented to determine the internal crack distribution by observing cross-sections of cracks that appear on plane surfaces of the material by means of the stereological principle. Only three types of surfaces are necessary for observation.

Key words: anisotropy, crack, fabric tensor, measurement, rock mass, stereology (IGC: F 0/F 3)

INTRODUCTION

Mechanical properties of a rock is greatly affected by the distribution of internal cracks. The crack distribution has two characteristics. One is the density of the cracks, say the total crack area per unit volume, and the other is its anisotropy, i. e., the degree to which the distribution is deviated from isotropic distribution. These characteristics must be expressed by particular quantities, if we want to incorporate them into constitutive equations which describe mechanical properties like fracture strength and macroscopic elastic moduli. Apparently, the density is described by a scalar quantity, while the anisotropy is described by a tensor quantity, which Oda (1982, 1983), Oda et al (1984) and Kanatani (1984 a, 1984 b) called the "fabric tensor." Oda (1982, 1983) and Oda et al (1984) also investigated the correlation

between the fabric tensor and mechanical properties like fracture strength and macroscopic elastic moduli.

When we apply this type of constitutive equations to real rocks, we must first know the internal crack distribution, which is very difficult to measure in practice. What can be observed is almost always restricted to the material surface. Hence, we must estimate the internal crack distribution from the data observed on the surface. This sort of study, i. e., estimating the three dimensional structure by two dimensional observations, has been studied as "integral geometry" in mathematics and known as "stereology" to people in metallurgy, biology and medicine (Santalo, 1953, 1976; Kendall and Moran, 1963; Elias, 1967; DeHoff and Rhines, 1968; Underwood, 1970; Miles and Serra, 1979; Weibel, 1979, 1980). In particular, estimation of the size distribution of

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particles inside a material from the size distribution of their cross-sections on the surface has been well studied (e. g., Goldsmith, 1967; Nicholson, 1970; Exner, 1972; Coleman, 1982, 1983; Cruz-Orive, 1983). The same principle applies to estimate the size distribution of internal cracks from the size distribution of their cross-sections on the surface (e. g., see Oda, 1983). Various computational schemes for it and numerical errors involved in them are also studied by Kanatani and Ishikawa (1985).

The estimation of anisotropy from observations on the surface has been also studied, dating back to "Buffon's needle problem" (Buffon, 1777). The problem was formulated as estimation of the "distribution density" by Hilliard (1967) and as estimation of the "fabric tensor" by Kanatani (1984 b). The same principle is used in computer vision and image processing (Kanatani, 1984 c, 1985 a). If we want to apply these results directly, we must cut the material by a large number of cutting planes and observe the resulting surfaces, which is impossible if the number of material samples are limited. However, if the anisotropy is known to be "weak," estimation schemes requiring only three types of cross-sections are available (Kanatani, 1985 b). In this paper, we present such a scheme of estimating the crack distribution based on the general theory of Kanatani (1985 b).

DISTRIBUTION DENSITY AND FABRIC TENSORS

In the following, we regard a crack as a plane surface without thickness. A non-planar crack is treated as an assembly of several cracks, each being planar. Attach a unit normal vector \mathbf{n} to each surface. Since there are two possibilities of opposite directions, choose one randomly with probability 1/2. (This is equivalent to dividing a crack area equally into two and attaching to them two normals of opposite directions.) Define the "distribution density" $f(\mathbf{n})$ in such a way that $f(\mathbf{n})d\Omega(\mathbf{n})$ is the summed area,

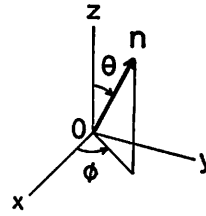


Fig. 1. The spherical coordinates associated with an xyz -coordinate system

per unit volume of the region under consideration, of those crack surfaces having normals inside the differential solid angle $d\Omega$ along \mathbf{n} . (The definition is the same as that of the distribution density of interparticle contacts of a granular material (Kanatani, 1981, 1983).

If the orientation of \mathbf{n} is described by its spherical coordinates θ and ϕ with respect to a Cartesian coordinate system fixed in the material, the distribution density $f(\mathbf{n})$ is also regarded as a function $f(\theta, \phi)$ of θ and ϕ , $0 \leq \theta < \pi$, $0 \leq \phi < 2\pi$ (see Fig. 1.). By definition.

$$c = \int f(\mathbf{n})d\Omega(\mathbf{n}) = \int_0^{2\pi} \int_0^\pi f(\theta, \phi) \sin\theta d\theta d\phi \quad (1)$$

is the "area density", i. e., the total area of cracks per unit volume. Since $f(\mathbf{n})$ or $f(\theta, \phi)$ is regarded as a function on a unit surface, it can be expressed by the following spherical harmonics expansion (cf. Kanatani, 1984 a).

$$f(\theta, \phi) = \frac{c}{4\pi} \left[1 + \sum_{n=2}^{\infty} \left\{ \frac{1}{2} a_{n0} P_n(\cos\theta) + \sum_{m=1}^n P_n^m(\cos\theta) [a_{nm} \cos m\phi + b_{nm} \sin m\phi] \right\} \right], \quad (2)$$

$$\begin{aligned} \begin{bmatrix} a_{nm} \\ b_{nm} \end{bmatrix} &= \frac{2(2n+1)}{c} \frac{(n+m)!}{(n-m)!} \\ &\times \int_0^{2\pi} \int_0^\pi f(\theta, \phi) P_n^m(\cos\theta) \begin{bmatrix} \cos m\phi \\ \sin m\phi \end{bmatrix} \\ &\times \sin\theta d\theta d\phi. \end{aligned} \quad (3)$$

Here, $P_n(z)$ is the n th Legendre polynomial, $P_n^m(z)$ is the associated Legendre function,

and \sum' designates summation with respect to only even indices. Odd terms do not appear because $f(\mathbf{n})$ or $f(\theta, \phi)$ is "symmetric" with respect to the origin, i. e., $f(\mathbf{n}) = f(-\mathbf{n})$ or $f(\theta, \phi) = f(\pi - \theta, \phi + \pi)$. In terms of Cartesian tensors, this expansion is expressed in the following form (cf. Kanatani, 1984 a).

$$f(\mathbf{n}) = \frac{c}{4\pi} [1 + D_{ij}n_i n_j + D_{ijkl}n_i n_j n_k n_l + \dots]. \quad (4)$$

Throughout this paper, Einstein's summation convention over repeated indices is adopted. Here, D_{ij} , D_{ijkl} , ... are symmetric deviator tensors (see Kanatani, 1984 a) and $\mathbf{n} = (\sin\theta \cos\phi, \sin\theta \sin\phi, \cos\theta)$. Kanatani (1984 a) called $D_{i_1 \dots i_n}$ the n th "fabric tensor" of the distribution. It is expressed in terms of the spherical expansion coefficients a_{nm} and b_{nm} , $m = 0, 1, \dots, n$ (Kanatani, 1984 a). In particular, we have for $n=2$

$$(D_{ij}) = \begin{bmatrix} -\frac{1}{4}a_{20} + 3a_{22} & 3b_{22} & \frac{3}{2}a_{21} \\ 3b_{22} & -\frac{1}{4}a_{20} - 3a_{22} & \frac{3}{2}b_{21} \\ \frac{3}{2}a_{21} & \frac{3}{2}b_{21} & \frac{1}{2}a_{20} \end{bmatrix} \quad (5)$$

In terms of $f(\mathbf{n})$, Eq. (3) becomes

$$D_{i_1 \dots i_n} = \frac{2n+1}{2^n c} \binom{2n}{n} N_{(i_1 \dots i_n)}, \quad (6)$$

where $N_{i_1 \dots i_n}$ is the n th "moment tensor" of $f(\mathbf{n})$, i. e.,

$$N_{i_1 \dots i_n} = \int n_{i_1} \dots n_{i_n} f(\mathbf{n}) d\Omega(\mathbf{n}), \quad (7)$$

and $\{ \}$ designates the "deviator part" of a symmetric tensor (Kanatani, 1984 a). In particular, N_{ij} is related to c and D_{ij} by

$$c = N_{ii}, \quad (8)$$

$$D_{ij} = -\frac{15}{2c} N_{ij} - \frac{5}{2} \delta_{ij}, \quad (9)$$

where δ_{ij} is the Kronecker delta.

Among the fabric tensors, the most important one is D_{ij} , which is zero for isotropic distribution. Hence, it is a measure of "deviation from isotropy." Those of higher

ranks D_{ijkl} , etc. describe higher order fluctuations (Kanatani, 1984 a). If higher order fluctuations are neglected, the distribution $f(\mathbf{n})$ represents a smooth orthogonal anisotropy, having the principal axes of D_{ij} as the symmetry axes. The distribution takes local maxima or minima along them. The corresponding eigenvalues of D_{ij} are the ratios of the distribution to the isotropic distribution along these axes. Thus, D_{ij} has a clear geometrical meaning. On the other hand, Oda (1982, 1983) called the moment tensor of Eq. (7) the "fabric tensor." It is related to ours by Eq. (6) or Eqs. (8) and (9). Eq. (7) can be interpreted as follows. Let V be the volume of the region in question. Number the cracks in it from 1 though N . Let $\mathbf{n}^{(1)}, \dots, \mathbf{n}^{(N)}$ be their respective unit normals and let $S^{(1)}, \dots, S^{(N)}$ be their respective areas. Then,

$$N_{i_1 \dots i_n} = \frac{1}{V} \sum_{a=1}^N n_{i_1}^{(a)} \dots n_{i_n}^{(a)} S^{(a)}. \quad (10)$$

If the crack distribution is nearly isotropic and higher order fluctuations can be neglected, it is sufficient to know the area density c and the fabric tensor D_{ij} to estimate the distribution density $f(\mathbf{n})$. Since c and D_{ij} are related to N_{ij} by Eqs. (8) and (9), it seems that we could compute N_{ij} by Eq. (10). However, it is very difficult to measure the orientations of the cracks inside a rock. Of course, we can measure the orientations of the cracks near a surface by careful observation, but the process is tedious. In what follows, we give a procedure to estimate c and D_{ij} from observations on surfaces by means of the stereological principle. This method would also be applicable to photographs of material surfaces, and hence it could lead to the use of computer image processing techniques in the future.

PRINCIPLE OF STEREOLOGICAL ESTIMATION

Imagine the following "stereological measurement." Place a line of orientation \mathbf{n} randomly in a rock material and count the number of intersections with the cracks in

it. Let $N(\mathbf{n})$, or $N(\theta, \phi)$ in terms of polar coordinates, be the observed data, namely the number of intersections per unit length of the probe line. Like the distribution density $f(\mathbf{n})$ or $f(\theta, \phi)$, it can be expressed as the following spherical harmonics expansion.

$$N(\theta, \phi) = \frac{C}{4\pi} \left[1 + \sum_{n=2}^{\infty} \left\{ \frac{1}{2} A_{n0} P_n(\cos\theta) + \sum_{m=1}^n P_n^m(\cos\theta) [A_{nm} \cos m\phi + B_{nm} \sin m\phi] \right\} \right], \quad (11)$$

$$C = \int_0^{2\pi} \int_0^\pi N(\theta, \phi) \sin\theta d\theta d\phi, \quad (12)$$

$$\begin{aligned} \left[\frac{A_{nm}}{B_{nm}} \right] &= \frac{2(2n+1)}{C} \frac{(n+m)!}{(n-m)!} \\ &\times \int_0^{2\pi} \int_0^\pi N(\theta, \phi) P_n^m(\cos\theta) \begin{bmatrix} \cos m\phi \\ \sin m\phi \end{bmatrix} \\ &\times \sin\theta d\theta d\phi. \end{aligned} \quad (13)$$

Odd terms do not appear because of the "symmetry" with respect to the origin, i.e., $N(\mathbf{n}) = N(-\mathbf{n})$ or $N(\theta, \phi) = N(\pi - \theta, \phi + \pi)$. The tensor form is

$$N(\mathbf{n}) = \frac{C}{4\pi} [1 + F_{i_1} n_{i_1} n_j + F_{i_1 j k l} n_{i_1} n_j n_k n_l + \dots], \quad (14)$$

and $F_{i_1 \dots i_n}$ is related to $N(\mathbf{n})$ in the same manner as $D_{i_1 \dots i_n}$ is to $f(\mathbf{n})$ by Eqs. (6) and (7). In particular, F_{i_j} is related to A_{20}, \dots, B_{22} by the same way as D_{i_j} is related to a_{20}, \dots, b_{22} by Eq. (5).

Now, if $N(\mathbf{n})$ or $N(\theta, \phi)$ are given by Eq. (14) or (11), the distribution density is given as follows (Kanatani, 1984 b).

$$f(\theta, \phi) = \frac{C/2\pi}{4\pi} \left[1 + \sum_{n=2}^{\infty} \lambda_n \left\{ \frac{1}{2} A_{n0} P_n(\cos\theta) + \sum_{m=1}^n P_n^m(\cos\theta) [A_{nm} \cos m\phi + B_{nm} \sin m\phi] \right\} \right], \quad (15)$$

$$\lambda_n = (-1)^{n/2-1} 2^{n-1} (n-1)(n+1) \binom{n}{n/2}. \quad (16)$$

The tensor form becomes

$$f(\mathbf{n}) = \frac{C/2\pi}{4\pi} [1 + 4F_{i_j} n_{i_1} n_j - 24F_{i_1 j k l} n_{i_1} n_j n_k n_l + \dots]. \quad (17)$$

Hence, once C and F_{i_j} are computed from the observed data $N(\mathbf{n})$ or $N(\theta, \phi)$, the area density c and the fabric tensor D_{i_j} are obtained by

$$c = C/2\pi, \quad D_{i_j} = 4F_{i_j}, \quad (18)$$

A most straightforward way to compute C and F_{i_j} is to approximate the integrals of Eqs. (12) and (13) by appropriate summations. For details, see Kanatani (1984 b). In order to do so, however, we must cut the material with planes of various orientations. This means we must prepare a large number of material samples and cut them with planes of various orientations. This becomes difficult when the amount of the sample material is limited. Here, we seek ways of estimation requiring only a small number of orientations of the cutting planes. This is possible if high order fluctuations of the distribution $f(\mathbf{n})$ can be neglected.

PROCEDURE OF STEREOLOGICAL ESTIMATION

Suppose high order fluctuations of the distribution density $f(\mathbf{n})$ can be neglected and $f(\mathbf{n})$ is described by

$$f(\mathbf{n}) = \frac{c}{4\pi} [1 + D_{i_j} n_{i_1} n_j]. \quad (19)$$

If this is the case, we say that the anisotropy of the crack distribution is "weak." If the anisotropy is weak, the observed data $N(\mathbf{n})$ becomes by Eq. (15)

$$N(\mathbf{n}) = \frac{C}{4\pi} [1 + F_{i_j} n_{i_1} n_j]. \quad (20)$$

This expression has 6 parameters, 1 for C and 5 for F_{i_j} (a symmetric deviator tensor). Hence, we need, in principle, only 6 data to determine them. However, since actual measurement always involves random errors, the necessary data must have the form of sums or averages of a large number of observations in order to cancel out possible random errors.

Now, consider the following integrations.

$$M^{(2)} = \int_{C(z)} N(\mathbf{n}) ds, \quad (21)$$

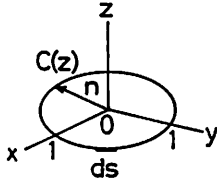


Fig. 2. Unit circle around the z -axis on the xy -plane

$$M_{ij}^{(z)} = \int_{C(z)} n_i n_j N(n) ds. \quad (22)$$

Here, $C(z)$ is a unit circle on the xy -plane around the z -axis, and $\int_{C(z)} ds$ designates the line integral along $C(z)$ (Fig. 2). If we substitute Eq. (20) in these, noting $F_{ij}=0$, we obtain the following expressions (For a more general case, see Kanatani, 1985 b).

$$\bar{M}^{(z)} = \frac{C}{2} \left(1 - \frac{1}{2} F_{33} \right), \quad (23)$$

$$M_{12}^{(z)} = \frac{C}{8} F_{12}. \quad (24)$$

Consider the same quantities defined on the yz - and zx -planes as well. Then, as is easily seen, we can obtain C and F_{ij} in terms of them as follows.

$$C = \frac{2}{3} (M^{(x)} + M^{(y)} + M^{(z)}), \quad (25)$$

$$F_{11} = \frac{2(-2M^{(x)} + M^{(y)} + M^{(z)})}{(M^{(x)} + M^{(y)} + M^{(z)})}, \quad (26)$$

$$F_{22} = \frac{2(M^{(x)} - 2M^{(y)} + M^{(z)})}{(M^{(x)} + M^{(y)} + M^{(z)})}, \quad (27)$$

$$F_{33} = \frac{2(M^{(x)} + M^{(y)} - 2M^{(z)})}{(M^{(x)} + M^{(y)} + M^{(z)})}, \quad (28)$$

$$F_{12} = \frac{12M_{12}^{(z)}}{(M^{(x)} + M^{(y)} + M^{(z)})} (=F_{21}), \quad (29)$$

$$F_{23} = \frac{12M_{23}^{(x)}}{(M^{(x)} + M^{(y)} + M^{(z)})} (=F_{32}), \quad (30)$$

$$F_{31} = \frac{12M_{31}^{(y)}}{(M^{(x)} + M^{(y)} + M^{(z)})} (=F_{13}). \quad (31)$$

These relations can be used for estimation of C and F_{ij} . Cut the material randomly with a plane parallel to the xy -plane and draw on it a line making angle $k\pi/N$, $k=0, 1, \dots, N-1$ from the x -axis (Fig. 3).

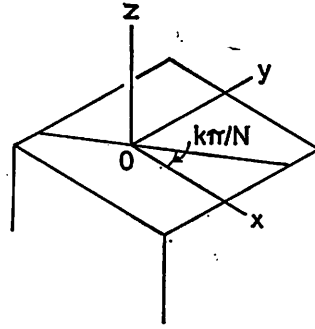


Fig. 3 A probe line drawn at angle $k\pi/N$ from the x -axis on a surface parallel to the xy -plane

Let $N_k^{(z)}$ be the number of intersections with the cracks per unit length of the line. Then, approximate $M^{(z)}$ and $M_{ij}^{(z)}$ by

$$M^{(z)} = 2\pi \sum_{k=0}^{N-1} N_k^{(z)} / N, \quad (32)$$

$$M_{12}^{(z)} = \pi \sum_{k=0}^{N-1} N_k^{(z)} \sin(2\pi k/N) / N. \quad (33)$$

Do the same thing for the yz - and the zx -plane and compute $M^{(x)}$, $M_{23}^{(x)}$, $M^{(y)}$ and $M_{31}^{(y)}$. Then, C and F_{ij} are given by Eqs. (25)–(31). The area density c and the fabric tensor D_{ij} are given by Eqs. (18).

Thus, we need to cut the material with only three different planes and hence to prepare only three material samples. If a rectangular box-shaped sample is available, we need to observe only its three faces. Of course, the accuracy is improved if we cut the material with equidistant parallel planes of each orientation and draw on it parallel lines with the separation equal to that of the parallel planes. Then, the average over the observations is taken, the length of each line being the weight.

CONCLUDING REMARKS

In this paper, we have shown a procedure to estimate the internal crack distribution of a rock mass from observations on its surface planes. An important idea behind this is the concept of the "distribution density" $f(n)$ or $f(\theta, \phi)$ which characterizes

the distribution of internal cracks. Also important is the fact that the distribution density $f(n)$ is described by the "area density" c and the "fabric tensors" D_{ij} , D_{ijkl} , etc.. The second rank fabric tensor D_{ij} is the most important one, because it describes the degree of deviation from isotropic distribution. Since any constitutive equations of rock materials must be tensor equations, it is expected that they incorporate the effects due to the crack distribution in terms of c and D_{ij} . Our method of stereological estimation directly determines c and D_{ij} from observations on surfaces when higher order effects can be neglected. We employed the count of intersections between probe lines and crack cross-sections, and this seems most practical in actual experiments. However, we could have also used the length of crack cross-sections which appear on a plane surface of a given orientation. For details, see Kanatani (1985 b).

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