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NOTE

Computational Cross Ratio for Computer Vision

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A new "computational" formulation of cross ratio is presented with a view to applications to computer vision problems by extending the framework of "computational projective geometry" of Kanatani (*Image Understand.* 54, 1991, 333–348). As typical examples, we construct procedures for computing the 3-D orientation of a planar shape from its 2-D projection image and the focus of expansion from an image trajectory of a single point by taking advantage of the perspective invariance of cross ratio and "projective coordinates," and the resulting 3-D interpretation of "harmonic range." © 1994 Academic Press, Inc.

1. INTRODUCTION

Cross ratio is the most essential invariant with respect to projective transformations, and its perspective invariance has attracted attention of computer vision researchers with a view to applications to object recognition from perspective images [2, 3, 7]. Today, interests in algebraic treatment of projective geometry and the use of *algebraic invariants* for object recognition is rapidly increasing [1, 4, 9–12].

However, all existing attempts to make use of the invariance of cross ratio have used "image coordinates" for computing cross ratio. This is quite natural since cross ratio plays the role of associating a metric space equipped with coordinates with the interpretation as an abstract *projective space* [13]. In real computation, however, the use of image coordinates will cause computational breakdown or deterioration of accuracy if they become too large (or infinite), although the cross ratio itself has a small magnitude. In fact, the advantage of using cross ratio lies in the very fact that *its value is kept invariant however large the magnitude of the individual image coordinates are.* How should we take advantage of this fact in real numerical computation?

In this paper, we give cross ratio an alternative "computational" form. Following the formulation of Kanatani [5], which he calls *computational projective geometry,* we represent all points and lines on the image plane by unit vectors called *N-vectors*. The use of N-vectors is equivalent to using (normalized) homogeneous coordinates by regarding the image plane as a projective space. As a result, points at infinity and the line at infinity can be treated in completely the same way as finite points and lines. This paper is a supplement to the computational projective geometry of Kanatani [5], giving cross ratio a computational formulation.

The merit of our formulation is not limited to merely rewriting existing formulas in a computationally favorable way. Our formulation sheds new light on many other applications. As typical examples, we write procedures for computing the 3-D orientation of a planar shape from its 2-D projection image and the focus of expansion from an image trajectory of a single point by taking advantage of the perspective invariance of cross ratio and *projective coordinates*, and the resulting 3-D interpretation of *harmonic range*.

2. COMPUTATIONAL PROJECTIVE GEOMETRY

2.1. *Perspective Transformation and N-Vectors*

Assume the following camera imaging model. The camera is associated with an XYZ coordinate system with origin O at the center of the lens and the Z axis along the optical axis (Fig. 1). The plane $Z = f$ is identified with the image plane, on which an xy image coordinate system is defined so that the x and the y axes are parallel to the X and the Y axes, respectively. Let us call the origin O the *viewpoint* and the constant f the *focal length*.

Any point (x, y) on the image plane is represented by the unit vector $\mathbf{m}$ indicating the orientation of the ray starting from the viewpoint O and passing through that point; any line $Ax + By + C = 0$ on the image plane is represented by the unit surface normal $\mathbf{n}$ to the plane passing through the viewpoint O and intersecting the image plane along that line (Fig. 1). Their components are given by

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$$\mathbf{m} = \pm N\left[\begin{pmatrix} x \\ y \\ f \end{pmatrix}\right], \quad \mathbf{n} = \pm N\left[\begin{pmatrix} A \\ B \\ C/f \end{pmatrix}\right], \tag{1}$$

where $N[\cdot]$ denotes normalization into a unit vector. Let us call $\mathbf{m}$ and $\mathbf{n}$ the *N-vectors* of the point and the line [5].

A point (X, Y, Z) in the scene is perspectively projected onto a point (x, y) on the image plane given by

$$x = f\frac{X}{Z}, \quad y = f\frac{Y}{Z}. \tag{2}$$

We define the N-vector of a point in the scene to be the N-vector of its projection on the image plane, and the N-vector of a line in the scene to be the N-vector of its projection on the image plane.

In order to avoid the confusion as to whether we are referring to a point in the scene or its projection on the image plane, we call a point in the scene a *space point* and a point on the image plane an *image point*. Similarly, we call a line in the scene a *space line* and a line on the image plane an *image line*. We then express the perspective projection relationship between a space point and its projection by such expressions as "a space point and the corresponding image point" and "an image line and the corresponding space line."

2.2. *Incidence Relation*

If $\mathbf{m}$ and $\mathbf{n}$ are the N-vectors of an image point P and an image line l, respectively, it is immediately seen that image point P is *on* image line l, or image line l *passes through* image point P, if and only if

$$(\mathbf{m}, \mathbf{n}) = 0, \tag{3}$$

where $(\cdot, \cdot)$ denotes the inner product of vectors. If this is the case, we also say that image point P and image line l are *incident* to each other.

An image point that is on two distinct image lines is called their *intersection*; an image line that passes through two distinct image points is called their *join*. If $\mathbf{n}_1$ and $\mathbf{n}_2$ are the N-vectors of two distinct image lines, the N-vector $\mathbf{m}$ of their intersection is given by

$$\mathbf{m} = \pm N[\mathbf{n}_1 \times \mathbf{n}_2], \tag{4}$$

because $\mathbf{m}$ must satisfy the incidence relation of Eq. (3) for both image lines: $(\mathbf{m}, \mathbf{n}_1) = 0$ and $(\mathbf{m}, \mathbf{n}_2) = 0$. Dually, if $\mathbf{m}_1$ and $\mathbf{m}_2$ are the N-vectors of two distinct image points, the N-vector $\mathbf{n}$ of their join is given by

$$\mathbf{n} = \pm N[\mathbf{m}_1 \times \mathbf{m}_2], \tag{5}$$

because $\mathbf{n}$ must satisfy the incidence relation of Eq. (3) for both image points: $(\mathbf{m}_1, \mathbf{n}) = 0$ and $(\mathbf{m}_2, \mathbf{n}) = 0$.

The use of N-vectors for representing image points and image lines is equivalent to using the *homogeneous coordinates*. Although homogeneous coordinates can be multiplied by any nonzero number from a mathematical point of view, this causes computational problems, so it is the most convenient to normalize them into a unit vector. Kanatani [5] reformulated projective geometry from this viewpoint, rewriting relationships of projective geometry as *computational procedures*. He called the resulting formulation *computational projective geometry*. In the following, we adopt his formulation and regard a unit vector $\mathbf{m}$ whose Z component is 0 as the N-vector of an ideal point (a *point at infinity*) and $\mathbf{n} = (0, 0, \pm 1)$ as the N-vector of the ideal line (the *line at infinity*).

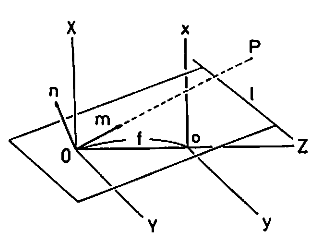


FIG. 1. Camera imaging geometry and N-vectors of a point and a line.

2.3. *Collineations*

Points are *collinear* if they are all on a common line; lines are *concurrent* if they all meet at a common point. A *collineation* is a one-to-one mapping between image points (including ideal points) and between image lines (including the ideal line) such that (i) collinear image points are mapped to collinear image points, (ii) concurrent image lines are mapped to concurrent image lines, and (iii) the incidence relation is preserved (i.e., if an image point (or line) is on (or passes through) an image line (or point), the mapped image point (or line) is on (or passes through) the mapped image line (or point)). It can be proved that this mapping is written in terms of N-vectors as follows: an image point of N-vector $\mathbf{m}$ is mapped to an image point of N-vector $\mathbf{m}'$, and an image line of N-vector $\mathbf{n}$ to an image line of N-vector $\mathbf{n}'$ in the form

$$\mathbf{m}' = \pm N[\mathbf{A}^\top \mathbf{m}], \quad \mathbf{n}' = \pm N[\mathbf{A}^{-1}\mathbf{n}], \tag{6}$$

where $\mathbf{A}$ is a nonsingular matrix, and $\top$ denotes transpose (see [6] for details). Since the matrix $\mathbf{A}$ is unique up to scale, we hereafter adopt the convention that $\mathbf{A}$ is scaled so that $\det \mathbf{A} = 1$. For simplicity, let us call the collineation represented by matrix $\mathbf{A}$ simply "collineation $\mathbf{A}$." In terms of *inhomogeneous coordinates* (i.e., image coordinates), the first of Eq. (6) for $\mathbf{A} = (A_{ij})$, $i, j = 1, 2, 3$, is rewritten as

$$x' = f\frac{A_{11}x + A_{21}y + A_{31}f}{A_{13}x + A_{23}y + A_{33}f}, \quad y' = f\frac{A_{12}x + A_{22}y + A_{32}f}{A_{13}x + A_{23}y + A_{33}f}. \tag{7}$$

As can be seen from Eq. (6), the mapping rule for N-vectors of image points is different from that for N-vectors of image lines. This is a consequence of the requirement that the incidence relation be preserved. Namely, N-vectors $\mathbf{m}$ and $\mathbf{n}$ such that $(\mathbf{m}, \mathbf{n}) = 0$ must be mapped to N-vectors $\mathbf{m}'$ and $\mathbf{n}'$ such that $(\mathbf{m}', \mathbf{n}') = 0$. This fact is expressed by saying that the mapping of image points and the mapping of image lines are *contragradient* to each other. We can also say that a vector mapped as an N-vector of an image point is a *contravariant vector,* while a vector mapped as an N-vector of an image line is a *covariant vector* [13].

It is easy to see that the composition of collineation $\mathbf{A}_1$ followed by collineation $\mathbf{A}_2$ coincides with collineation $\mathbf{A}_1\mathbf{A}_2$ (and $\det(\mathbf{A}_1\mathbf{A}_2) = \det \mathbf{A}_1 \det \mathbf{A}_2 = 1$). It is also easy to see that the inverse of collineation $\mathbf{A}$ is given by collineation $\mathbf{A}^{-1}$ (and $\det \mathbf{A}^{-1} = 1/\det \mathbf{A} = 1$). Evidently, the identity mapping is a trivial collineation represented by the unit matrix $\mathbf{I}$ (and $\det \mathbf{I} = 1$). Thus, the set of all collineations forms a group, called the *group of 2-D projective transformations,* which is isomorphic to SL(3)—the group of three-dimensional matrices of determinant 1 under matrix multiplication. A collineation is also called a *projective transformation* or simply *projectivity*.

2.4. *Vanishing Points and Vanishing Lines*

As is well known, projections of parallel space lines meet at a common "vanishing point" on the image plane. Formally, the vanishing point of a space line is the limit of the projection of a point that moves along the space line indefinitely in one direction (both directions define the same vanishing point). From Fig. 2a, it is easy to confirm the following theorem:

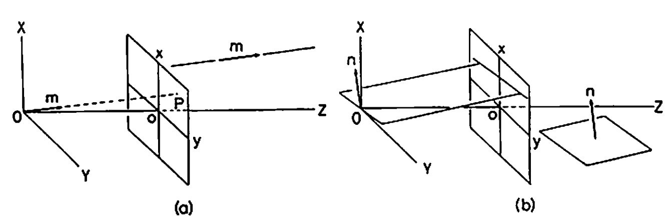


FIG. 2. (a) The vanishing point of a space line. (b) The vanishing line of a planar surface in the scene.

THEOREM 1. *A space line extending along a unit vector* $\mathbf{m}$ *has, when projected, a vanishing point of N-vector* $\pm\mathbf{m}$.

Since the vanishing point is determined by the 3-D orientation of the space line alone irrespective of its location in the scene, we see that:

COROLLARY 1. *Projections of parallel space lines intersect at a common vanishing point.*

As is also well known, projections of planar surfaces that are parallel in the scene define a common "vanishing line." Formally, the *vanishing line* of a planar surface in the scene is the set of all the vanishing points of space lines lying on the surface. From Fig. 2b, it is easy to confirm the following theorem:

THEOREM 2. *A planar surface of unit surface normal* $\mathbf{n}$ *has, when projected, a vanishing line of N-vector* $\pm\mathbf{n}$.

Since the vanishing line is determined by the 3-D orientation of the planar surface alone, irrespective of its location in the scene, we see that:

COROLLARY 2. *Projections of planar surfaces that are parallel in the scene define a common vanishing line.*

In summary, if a vanishing point is detected on the image plane, its N-vector indicates the 3-D orientation of the space line, and if a vanishing line is detected on the image plane, its N-vector indicates the surface normal to the planar surface. This 3-D interpretation of vanishing points and vanishing lines play an essential role in 3-D scene analysis of machine vision.

If we use Eqs. (4) and (5) for computing intersections and joins, computation of vanishing points and vanishing lines causes no computational problems wherever the vanishing points and vanishing lines appear, even at infinity (see [5, 6] for details of optimal estimation).

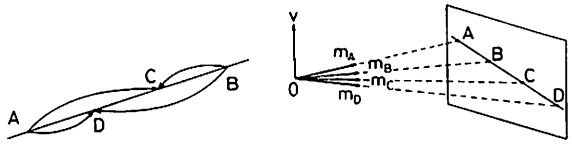
3. CROSS RATIO AND PROJECTIVE COORDINATES

3.1. *Perspective Invariance of Cross Ratio*

Let A, B, C, and D be distinct points on line l. Their *cross ratio* (or *anharmonic ratio*) $[ABCD]$ is defined by

$$[ABCD] = \frac{AC}{BC} \Big/ \frac{AD}{BD}, \tag{8}$$

where AC, BC, etc. are *signed* distances with respect to an arbitrarily fixed orientation of the line l (hence, $CA = -AC$, etc.; Fig. 3a). From this definition, the following relationships are obvious:



(a) (b)

FIG. 3. (a) The cross ratio of four collinear image points. (b) Computation of cross ratio in terms of N-vectors.

$$[ABCD] = [BADC] = [CDAB] = [DCBA],$$
$$[ABDC] = \frac{1}{[ABCD]}, \quad [ACBD] = 1 - [ABCD],$$
$$[ACDB] = \frac{1}{1-[ABCD]}, \quad [ADBC] = \frac{[ABCD]-1}{[ABCD]}, \tag{9}$$
$$[ADCB] = \frac{[ABCD]}{[ABCD]-1}.$$

This is the original definition of cross ratio, but this form is not convenient for actual computation. Let $|\mathbf{a}, \mathbf{b}, \mathbf{c}|$ $(= (\mathbf{a} \times \mathbf{b}, \mathbf{c}) = (\mathbf{b} \times \mathbf{c}, \mathbf{a}) = (\mathbf{c} \times \mathbf{a}, \mathbf{b}))$ denote the scalar triple product of vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$. We now show that the cross ratio of image points can be computed in terms of their N-vectors as follows:

Proposition 1. *If $\mathbf{m}_A$, $\mathbf{m}_B$, $\mathbf{m}_C$, and $\mathbf{m}_D$ are the N-vectors of collinear image points A, B, C, and D, respectively, and if $\mathbf{n}$ is the N-vector of the image line l on which these points lie, the cross ratio $[ABCD]$ is given by*

$$[ABCD] = \frac{|\mathbf{m}_A, \mathbf{m}_C, \mathbf{v}|}{|\mathbf{m}_B, \mathbf{m}_C, \mathbf{v}|} \Big/ \frac{|\mathbf{m}_A, \mathbf{m}_D, \mathbf{v}|}{|\mathbf{m}_B, \mathbf{m}_D, \mathbf{v}|}, \tag{10}$$

where $\mathbf{v}$ is an arbitrary vector such that $(\mathbf{v}, \mathbf{n}) \neq 0$.

Proof. Let h be the distance of the image line l from the viewpoint O. Consider the triangle ΔOAC (Fig. 3b). Its area equals $\|\vec{OA} \times \vec{OC}\|/2$ and also $|AC| \cdot h/2$. Similarly, $\|\vec{OB} \times \vec{OC}\|/2 = |BC| \cdot h/2$, $\|\vec{OA} \times \vec{OD}\|/2 = |AD| \cdot h/2$, and $\|\vec{OB} \times \vec{OD}\|/2 = |BD| \cdot h/2$. Hence,

$$|AC| = \frac{1}{h}\|\vec{OA} \times \vec{OC}\|, \quad |BC| = \frac{1}{h}\|\vec{OB} \times \vec{OC}\|,$$
$$|AD| = \frac{1}{h}\|\vec{OA} \times \vec{OD}\|, \quad |BD| = \frac{1}{h}\|\vec{OB} \times \vec{OD}\|. \tag{11}$$

Vectors $\vec{OA} \times \vec{OC}$, $\vec{OB} \times \vec{OC}$, $\vec{OA} \times \vec{OD}$, and $\vec{OB} \times \vec{OD}$ are all parallel to the N-vector $\mathbf{n}$ of the image line l. Hence, if $(\mathbf{v}, \mathbf{n}) \neq 0$, the following relation holds:

$$\frac{(\vec{OA} \times \vec{OC}, \mathbf{v})}{AC} = \frac{(\vec{OB} \times \vec{OC}, \mathbf{v})}{BC} = \frac{(\vec{OA} \times \vec{OD}, \mathbf{v})}{AD} = \frac{(\vec{OB} \times \vec{OD}, \mathbf{v})}{BD}. \tag{12}$$

Since $(\vec{OA} \times \vec{OC}, \mathbf{v}) = |\vec{OA}, \vec{OC}, \mathbf{v}|$, etc., we have

$$[ABCD] = \frac{|\vec{OA}, \vec{OC}, \mathbf{v}|}{|\vec{OB}, \vec{OC}, \mathbf{v}|} \Big/ \frac{|\vec{OA}, \vec{OD}, \mathbf{v}|}{|\vec{OB}, \vec{OD}, \mathbf{v}|}. \tag{13}$$

Since $\vec{OA}$ is parallel to $\mathbf{m}_A$, we can write $\vec{OA} = c_A \mathbf{m}_A$ for a nonzero constant c_A. Similarly, $\vec{OB} = c_B \mathbf{m}_B$, $\vec{OC} = c_C \mathbf{m}_C$, and $\vec{OD} = c_D \mathbf{m}_D$ for nonzero constants c_B, c_C, and c_D. If these are substituted into the above expression, the constants, c_A, c_B, c_C, and c_D are all canceled out, resulting in Eq. (10). ■

Cross ratio can also be defined for any four collinear space points by Eq. (8). Since the cross ratio can be expressed solely in terms of N-vectors, we immediately obtain the following *perspective invariance* of cross ratio:

Corollary 3. *The cross ratio of four collinear space points is equal to the cross ratio of their corresponding image points.*

Points on a plane are *in general position* if no three of them are collinear; lines on a plane are in general position if no three of them are concurrent. Corollary 3 can be used to compute the 3-D position and orientation of a planar surface in the scene from its projection by identifying four space points in general position on it, provided the exact geometry (i.e., the distance of each pair) is known on the surface.

Let $\bar{A}$, $\bar{B}$, $\bar{C}$, and $\bar{D}$ be four space points in general position on a planar surface with a known geometry. Let A, B, C, and D be their corresponding image points, and $\mathbf{m}_A$, $\mathbf{m}_B$, $\mathbf{m}_C$, and $\mathbf{m}_D$ their respective N-vectors. The unit normal $\mathbf{n}$ to the surface and its distance d from the viewpoint O can be computed by the following procedure:

procedure *surface* $(A, B, C, D; \bar{A}, \bar{B}, \bar{C}, \bar{D})$

1. Compute the N-vectors $\mathbf{n}_{AC} = \pm N[\mathbf{m}_A \times \mathbf{m}_C]$ and $\mathbf{n}_{BD} = \pm N[\mathbf{m}_B \times \mathbf{m}_D]$ of the joints AC and BD, respectively (Eq. (5)).
2. Computer the N-vector $\mathbf{m}_I = \pm N[\mathbf{n}_{AC} \times \mathbf{n}_{BD}]$ of the intersection of AC and BD (Eq. (4)).
3. Computer the N-vectors $\mathbf{m}_R$ and $\mathbf{m}_S$ of R and S, respectively, where R is an image point on AC, and S an image point on BD, such that $[RAIC] = \bar{A}\bar{C}/\bar{A}\bar{I}$ and $[SBID] = \bar{B}\bar{D}/\bar{D}\bar{I}$.
4. Return the unit surface normal $\mathbf{n} = \pm N[\mathbf{m}_R \times \mathbf{m}_S]$.
5. Return the distance d determined so that a pair of space points, say space points $\bar{I}$ and $\bar{A}$ (or any other pair), has the prescribed distance.

Derivation. See Fig. 4. Let $\bar{I}$ be the space point on the surface corresponding to image point I. Let $\bar{R}$ be a space point on $\bar{A}\bar{C}$, and $\bar{S}$ a space point on $\bar{B}\bar{D}$. If R and S are their corresponding image points, the perspective invariance of cross ration (Proposition 1) implies that

$$[RAIC] = \frac{\bar{R}\bar{I}}{\bar{A}\bar{I}} \Big/ \frac{\bar{R}\bar{C}}{\bar{A}\bar{C}}, \quad [SBID] = \frac{\bar{S}\bar{I}}{\bar{B}\bar{I}} \Big/ \frac{\bar{S}\bar{D}}{\bar{A}\bar{D}}. \tag{14}$$

Move $\bar{R}$ on $\bar{A}\bar{C}$ indefinitely away from $\bar{I}$ (in whichever direction). Similarly, move $\bar{S}$ on $\bar{B}\bar{D}$ indefinitely away from $\bar{I}$. In the limit, we have $\bar{R}\bar{I}/\bar{R}\bar{C} \to 1$ and $\bar{S}\bar{I}/\bar{S}\bar{D} \to 1$, and hence

$$[RAIC] \to \frac{\bar{A}\bar{C}}{\bar{A}\bar{I}}, \quad [SBID] \to \frac{\bar{B}\bar{D}}{\bar{D}\bar{I}}. \tag{15}$$

(Recall that $\bar{R}\bar{I}$, $\bar{R}\bar{C}$, etc. are signed distances.) On the image plane, the corresponding image points R and S approach the vanishing points of $\bar{A}\bar{C}$ and $\bar{B}\bar{D}$, respectively. Hence, their join RS becomes the vanishing line of the surface. The N-vector $\mathbf{n}$ of the join RS is computed in terms of the N-vectors $\mathbf{m}_R$ and $\mathbf{m}_S$ of image points R and S by Eq. (5), and it indicates the 3-D orientation of the surface (Theorem 1).

This result is obvious in view of the perspective invariance of cross ratio and the 3-D interpretation of vanishing points and vanishing lines. However, our purpose here is not to show this fact itself but to construct the computational procedures of Steps 3 and 5.

Step 3 is computed as follows.

Proposition 2. *Let A, B, C, and D be distinct collinear image points on image line l of N-vector $\mathbf{n}$, and let $\mathbf{m}_A$, $\mathbf{m}_B$, $\mathbf{m}_C$, and $\mathbf{m}_D$ be their respective N-vectors. If the cross ratio $[ABCD]$ is specified, the N-vector of one, say D, is determined from the N-vectors of the rest in the form*

$$\mathbf{m}_D = \pm N[[ABCD]|\mathbf{m}_B, \mathbf{m}_C, \mathbf{n}|\mathbf{m}_A - |\mathbf{m}_A, \mathbf{m}_C, \mathbf{n}|\mathbf{m}_B]. \tag{16}$$

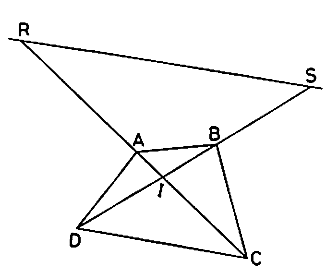


FIG. 4. The position and orientation of a planar surface can be computed if we can identify four space points in general position that have a known geometry on the surface.

Proof. Since image points A, B, and D are collinear, vector $\mathbf{m}_D$ is expressed as a linear combination of $\mathbf{m}_A$ and $\mathbf{m}_B$ in the form

$$\mathbf{m}_D = a\mathbf{m}_A + b\mathbf{m}_B. \tag{17}$$

If we note the relationships

$$\begin{aligned} |\mathbf{m}_A, \mathbf{m}_D, \mathbf{v}| &= |\mathbf{m}_A, a\mathbf{m}_A + b\mathbf{m}_B, \mathbf{v}| = b|\mathbf{m}_A, \mathbf{m}_B, \mathbf{v}|, \\ |\mathbf{m}_B, \mathbf{m}_D, \mathbf{v}| &= |\mathbf{m}_B, a\mathbf{m}_A + b\mathbf{m}_B, \mathbf{v}| = -a|\mathbf{m}_A, \mathbf{m}_B, \mathbf{v}|, \end{aligned} \tag{18}$$

Proposition 1 implies

$$\begin{aligned} [ABCD] &= \frac{|\mathbf{m}_A, \mathbf{m}_C, \mathbf{v}|}{|\mathbf{m}_B, \mathbf{m}_C, \mathbf{v}|} \Big/ \frac{|\mathbf{m}_A, \mathbf{m}_D, \mathbf{v}|}{|\mathbf{m}_B, \mathbf{m}_D, \mathbf{v}|} \\ &= -\frac{a}{b}\frac{|\mathbf{m}_A, \mathbf{m}_C, \mathbf{v}|}{|\mathbf{m}_B, \mathbf{m}_C, \mathbf{v}|}. \end{aligned} \tag{19}$$

Hence, $a = -b[ABCD]|\mathbf{m}_B, \mathbf{m}_C, \mathbf{v}|/|\mathbf{m}_A, \mathbf{m}_C, \mathbf{v}|$, and

$$\mathbf{m}_D = -b\left(\frac{|\mathbf{m}_B, \mathbf{m}_C, \mathbf{v}|}{|\mathbf{m}_A, \mathbf{m}_C, \mathbf{v}|}[ABCD] - \mathbf{m}_B\right). \tag{20}$$

The constant b is determined so that $\mathbf{m}_D$ is scaled into a unit vector. Namely,

$$\mathbf{m}_D = \pm N[[ABCD]|\mathbf{m}_B, \mathbf{m}_C, \mathbf{v}|\mathbf{m}_A - |\mathbf{m}_A, \mathbf{m}_C, \mathbf{v}|\mathbf{m}_B]. \tag{21}$$

Since $\mathbf{v}$ is arbitrary as long as $(\mathbf{v}, \mathbf{n}) \neq 0$, it can be taken to be $\mathbf{n}$ itself. ∎

In this form, no computational problems arise because all computation is done in terms of unit vectors: even if point D is at infinity, its N-vector $\mathbf{m}_D$ is correctly computed.

Step 5 is computed as follows.

Proposition 3. *Let A and B be two distinct space points on a planar surface in the scene, and let $\mathbf{m}_A$ and $\mathbf{m}_B$ be their N-vectors signed so that their Z components are positive. Let $\mathbf{n}$ be the unit surface normal to the surface. If the distance $|AB|$ is known, the distance d to the surface from the viewpint O is given by*

$$d = \frac{|AB| \cdot |(\mathbf{m}_A, \mathbf{n})| \cdot |(\mathbf{m}_B, \mathbf{n})|}{\|(\mathbf{m}_A, \mathbf{n})\mathbf{m}_B - (\mathbf{m}_B, \mathbf{n})\mathbf{m}_A\|}. \tag{22}$$

Proof. We can write $\vec{OA} = |OA|\mathbf{m}_A$ and $\vec{OB} = |OB|\mathbf{m}_B$. Since $\vec{AB} = \vec{OB} - \vec{OA}$ is orthogonal to $\mathbf{n}$, we have

$$(\vec{AB}, \mathbf{n}) = |OB|(\mathbf{m}_B, \mathbf{n}) - |OA|(\mathbf{m}_A, \mathbf{n}) = 0, \tag{23}$$

or $|OB| = |OA|(\mathbf{m}_A, \mathbf{n})/(\mathbf{m}_B, \mathbf{n})$. Hence,

$$\vec{AB} = |OB|\mathbf{m}_B - |OA|\mathbf{m}_A = \frac{|OA|}{(\mathbf{m}_B, \mathbf{n})}((\mathbf{m}_A, \mathbf{n})\mathbf{m}_B - (\mathbf{m}_B, \mathbf{n})\mathbf{m}_A). \quad (24)$$

Since the distance $|AB|$ is known, we have

$$|OA| = \frac{|AB| \cdot |(\mathbf{m}_B, \mathbf{n})|}{\|(\mathbf{m}_A, \mathbf{n})\mathbf{m}_B - (\mathbf{m}_B, \mathbf{n})\mathbf{m}_A\|}. \quad (25)$$

If we note that the distance d is given by $d = |(\vec{OA}, \mathbf{n})| = |OA| \cdot |(\mathbf{m}_A, \mathbf{n})|$, we obtain Eq. (22). ∎

We must point out an important fact: the problem described by the procedure *surface* (A, B, C, D; $\bar{A}$, $\bar{B}$, $\bar{C}$, $\bar{D}$) is *overconstrained*. If the true shape $\bar{A}\bar{B}\bar{C}\bar{D}$ is known, it cannot be projected onto an *arbitrary* shape $ABCD$. This is easily understood if we note that the position and orientation of $\bar{A}\bar{B}\bar{C}\bar{D}$ in the scene is specified by six parameters (three for position and three for orientation). Hence, the eight x and y image coordinates of A, B, C, and D cannot assume arbitrary values. This means that the above procedure is *fragile* in the sense that in the presence of noise it may produce different solutions if the four image points are labeled as A, B, C, and D differently.

There exists an alternative solution. If the collineation that maps the true geometry to the observed geometry is computed by the method given shortly, the 3-D motion between them is computed analytically [5, 8, 14–16]. This approach is *robust* in the sense that the solution is exact if noise is not present and also expected to be a good approximation in the presence of noise. However, this deviates from the main subject of this paper, so we omit the details (see [6] for the details).

3.2. *Projective Invariance of Cross Ratio*

Cross ratio is not only invariant to perspective projection; we can also prove its *projective invariance:*

THEOREM 3. *The cross ratio is invariant under collineations.*

Proof. Let A, B, C, and D be four image points on image line l, and $\mathbf{m}_A$ $\mathbf{m}_B$, $\mathbf{m}_C$, and $\mathbf{m}_D$ their respective N-vectors. If collineation $\mathbf{A}$ maps these points to A', B', C', and D', respectively, their N-vectors are given by

$$\mathbf{m}'_A = \gamma_A \mathbf{A}^\top \mathbf{m}_A, \quad \mathbf{m}'_B = \gamma_B \mathbf{A}^\top \mathbf{m}_B, \qquad \mathbf{m}'_C = \gamma_C \mathbf{A}^\top \mathbf{m}_C, \quad \mathbf{m}'_D = \gamma_D \mathbf{A}^\top \mathbf{m}_D, \quad (26)$$

where γ_A, γ_B, γ_C, and γ_D are normalization constants. Let $\mathbf{n}$ be the N-vector of image line l. Collineation $\mathbf{A}$ maps it into $\mathbf{n}' = c\mathbf{A}^{-1}\mathbf{n}$, where c is a normalization constant. Let $\mathbf{v}$ be a vector such that $(\mathbf{v}, \mathbf{n}) \neq 0$. Let $\mathbf{v}' = \mathbf{A}^\top\mathbf{v}$. Then,

$$(\mathbf{v}', \mathbf{n}') = (\mathbf{A}^\top\mathbf{v}, c\mathbf{A}^{-1}\mathbf{n}) = c(\mathbf{v}, \mathbf{A}\mathbf{A}^{-1}\mathbf{n}) = c(\mathbf{v}, \mathbf{n}) \neq 0. \quad (27)$$

According to PROPOSITION 1, the cross ratio of image points A', B', C', and D' is

$$[A'B'C'D'] = \frac{|\mathbf{m}'_A, \mathbf{m}'_C, \mathbf{v}'|}{|\mathbf{m}'_B, \mathbf{m}'_C, \mathbf{v}'|} \Big/ \frac{|\mathbf{m}'_A, \mathbf{m}'_D, \mathbf{v}'|}{|\mathbf{m}'_B, \mathbf{m}'_D, \mathbf{v}'|}. \quad (28)$$

Note the relationships

$$\begin{aligned} |\mathbf{m}'_A, \mathbf{m}'_C, \mathbf{v}'| &= \gamma_A\gamma_C|\mathbf{A}^\top\mathbf{m}_A, \mathbf{A}^\top\mathbf{m}_C, \mathbf{A}^\top\mathbf{v}| = \gamma_A\gamma_C \det \mathbf{A}^\top|\mathbf{m}_A, \mathbf{m}_C, \mathbf{v}|, \\ |\mathbf{m}'_B, \mathbf{m}'_C, \mathbf{v}'| &= \gamma_B\gamma_C|\mathbf{A}^\top\mathbf{m}_B, \mathbf{A}^\top\mathbf{m}_C, \mathbf{A}^\top\mathbf{v}| = \gamma_B\gamma_C \det \mathbf{A}^\top|\mathbf{m}_B, \mathbf{m}_C, \mathbf{v}|, \\ |\mathbf{m}'_A, \mathbf{m}'_D, \mathbf{v}'| &= \gamma_A\gamma_D|\mathbf{A}^\top\mathbf{m}_A, \mathbf{A}^\top\mathbf{m}_D, \mathbf{A}^\top\mathbf{v}| = \gamma_A\gamma_D \det \mathbf{A}^\top|\mathbf{m}_A, \mathbf{m}_D, \mathbf{v}|, \\ |\mathbf{m}'_B, \mathbf{m}'_D, \mathbf{v}'| &= \gamma_B\gamma_D|\mathbf{A}^\top\mathbf{m}_B, \mathbf{A}^\top\mathbf{m}_D, \mathbf{A}^\top\mathbf{v}| = \gamma_B\gamma_D \det \mathbf{A}^\top|\mathbf{m}_B, \mathbf{m}_D, \mathbf{v}|, \end{aligned} \quad (29)$$

If these are substituted into Eq. (28), constants γ_A, γ_B, γ_C, γ_D, and $\det \mathbf{A}^\top$ are all canceled out, yielding the expression of $[ABCD]$. ∎

Let P_∞, P_O, and P_I be three distinct points on line l. The (one-dimensional) *projective coordinate* $[P]$ of point P on line l with P_∞ the *supporting point*, P_O the *origin*, and P_I the *unit point* is defined by

$$[P] = [P_\infty P_O P_I P] \quad (30)$$

(Fig. 5a). If P coincides with any of P_∞, P_O, and P_I, the limit is taken. Evidently,

$$[P_\infty] = \pm\infty, \quad [P_O] = 0, \quad [P_I] = 1, \quad (31)$$

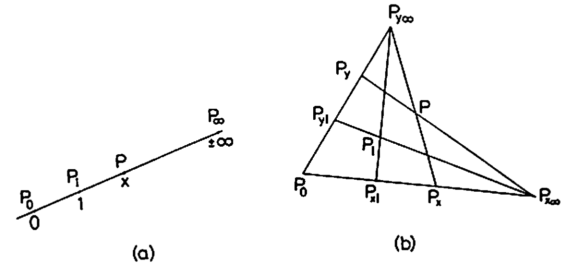


FIG. 5. (a) One-dimensional projective coordinate. (b) Two-dimensional projective coordinates.

where the sign of $[P_\infty]$ depends on which side the limit is taken from. For brevity, let us call $[P]$ the *coordinate* with respect to $\{P_\infty, P_O, P_I\}$.

This projective coordinate reduces to the ordinary coordinate (*Euclidean coordinate*) if P_∞ is the ideal point of line l located at infinity. Combining this fact and COROLLARY 3, we observe the following "3-D interpretation":

PROPOSITION 4. *Let $\bar{P}_O$ and $\bar{P}_I$ be distinct space points on a space line l, and P_O and P_I their corresponding image points. Let P_∞ be the vanishing point of the space line l. The coordinate of an image point with respect to $\{P_\infty, P_O, P_I\}$ coincides with the Euclidean coordinate of the corresponding space point on the space line l with $\bar{P}_O$ as the origin and $|\bar{P}_O\bar{P}_I|$ as unit length.*

Let $P_{x\infty}$, $P_{y\infty}$, P_O, and P_I be four coplanar points in general position. The two-dimensional *projective* coordinates $[P] = (x, y)$ of point P with $P_{x\infty}$ and $P_{y\infty}$ the supporting points, P_O the origin, and P_I the unit point are defined as follows. Let P_{xI} and P_x be the intersections of $P_OP_{x\infty}$ with $P_{y\infty}P_I$ and $P_{y\infty}P$, respectively (Fig. 5b). The x coordinate is defined by the cross ratio

$$x = [P_{x\infty}P_OP_{xI}P_x]. \quad (32)$$

Similarly, let P_{yI} and P_y be the intersections of $P_OP_{y\infty}$ with $P_{x\infty}P_I$ and $P_{x\infty}P$, respectively. The y coordinate is defined by the cross ratio

$$y = [P_{y\infty}P_OP_{yI}P_y]. \quad (33)$$

Evidently,

$$\begin{aligned} &[P_{x\infty}] = (\pm\infty, 0), \ [P_{y\infty}] = (0, \pm\infty), \\ &[P_O] = (0, 0), \ [P_I] = (1, 1). \end{aligned} \quad (34)$$

For brevity, let us call (x, y) the *coordinates* with respect to $\{P_{x\infty}, P_{x\infty}, P_O, P_I\}$.

These projective coordinates reduce to the ordinary Cartesian coordinates (Euclidean coordinates) if (i) $P_{x\infty}$ and $P_{y\infty}$ are ideal points located at infinity, (ii) $P_OP_{x\infty}$ and $P_OP_{y\infty}$ are orthogonal to each other, and (iii) $|P_OP_{xI}| = |P_OP_{yI}|$. Thus, we again observe the following 3-D interpretation.

PROPOSITION 5. *Let $\bar{P}_O$, $\bar{P}_{xI}$, and $\bar{P}_{yI}$ be distinct space points on a planar surface in the scene such that $\bar{P}_O\bar{P}_{xI} \perp \bar{P}_O\bar{P}_{yI}$ and $|\bar{P}_O\bar{P}_{xI}| = |\bar{P}_O\bar{P}_{yI}|$. Let P_O, P_{xI}, and P_{yI} be their corresponding image points. Let $P_{x\infty}$ and $P_{y\infty}$ be, respectively, the intersections of image lines P_OP_{xI} and P_OP_{yI} with the vanishing line of the surface. The coordinates (x, y) of an image point with respect to $\{P_{x\infty}, P_{y\infty}, P_O, P_I\}$ coincide with the Euclidean coordinates of the corresponding space point on the surface with $\bar{P}_O$ as the origin, $\bar{P}_O\bar{P}_{xI}$ and $\bar{P}_O\bar{P}_{yI}$ as the x and the y axes, respectively, and $|\bar{P}_O\bar{P}_{xI}| = |\bar{P}_O\bar{P}_{yI}|$ as unit length.*

The definition of two-dimensional projective coordinates and the projective invariance of cross ratio (THEOREM 3) imply the following:

PROPOSITION 6. *A unique collineation is determined that maps four arbitrarily given image points in general position to four arbitrarily given image points in general position.*

Proof. A two-dimensional projective coordinate system is defined by the first set of four image points by arbitrarily choosing them as the supporting points, the origin, and the unit point. Define the corresponding coordinate system from the second set of points. Due to the projective invariance of cross ratio (THEOREM 3), the collineation that maps the first set of image points to the second set maps an image point of coordinates (x, y) with respect to the first system to the image point of the same coordinates (x, y) with respect to the second system. ∎

PROPOSITION 7. *A unique collineation is determined that maps four arbitrarily given image lines in general position to four arbitrarily given image lines in general position.*

Proof. Four image lines in general position have six intersections, among which four image points in general position can be chosen. They define a two-dimensional projective coordinate system, and PROPOSITION 6 applies. ∎

The above facts are mere restatements of well known facts in projective geometry [13]. However, our interest is in actual computations. Let $(\mathbf{a}, \mathbf{b}, \mathbf{c})$ denote the matrix consisting of columns $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ in this order.

PROPOSITION 8. *Let $\mathbf{m}_{x\infty}$, $\mathbf{m}_{y\infty}$, $\mathbf{m}_O$, and $\mathbf{m}_I$ be the N-vectors of image points $P_{x\infty}$, $P_{y\infty}$, P_O, and P_I in general position, respectively. The collineation that maps image points $(\pm\infty, 0)$, $(0, \pm\infty)$, $(0, 0)$, and (f, f) to image points $P_{x\infty}$, $P_{y\infty}$, P_O, and P_I is given by the matrix*

$$\mathbf{A} = k(a\mathbf{m}_{x\infty}, b\mathbf{m}_{y\infty}, c\mathbf{m}_O)^\top, \quad (35)$$

where a, b, and c are defined by

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = (\mathbf{m}_{x\infty}, \mathbf{m}_{y\infty}, \mathbf{m}_O)^{-1}\mathbf{m}_I, \quad (36)$$

and k is a constant that scales $\mathbf{A}$ so that $\det \mathbf{A} = 1$.

Proof. The N-vectors of image points $(\pm\infty, 0)$, $(0, \pm\infty)$, $(0, 0)$, and (f, f) are, respectively,

$$\pm\begin{pmatrix}1\\0\\0\end{pmatrix}, \pm\begin{pmatrix}0\\1\\0\end{pmatrix}, \pm\begin{pmatrix}0\\0\\1\end{pmatrix}, \pm\begin{pmatrix}1/\sqrt{3}\\1/\sqrt{3}\\1/\sqrt{3}\end{pmatrix}. \tag{37}$$

If matrix $\mathbf{A}$ is defined by Eqs. (35) and (36), it is easy to see that

$$\pm N\left[\mathbf{A}^{\mathrm{T}}\begin{pmatrix}1\\0\\0\end{pmatrix}\right] = \pm\mathbf{m}_{x\infty}, \pm N\left[\mathbf{A}^{\mathrm{T}}\begin{pmatrix}0\\1\\0\end{pmatrix}\right] = \pm\mathbf{m}_{y\infty},$$

$$\pm N\left[\mathbf{A}^{\mathrm{T}}\begin{pmatrix}0\\0\\1\end{pmatrix}\right] = \pm\mathbf{m}_{O}, \tag{38}$$

$$\pm N\left[\mathbf{A}^{\mathrm{T}}\begin{pmatrix}1/\sqrt{3}\\1/\sqrt{3}\\1/\sqrt{3}\end{pmatrix}\right] = \pm N\left[(\mathbf{m}_{x\infty}, \mathbf{m}_{y\infty}, \mathbf{m}_{O})\begin{pmatrix}a\\b\\c\end{pmatrix}\right] = \pm\mathbf{m}_{I}. \tag{39}$$

■

From this proposition, we can obtain the following procedure for computing the coordinates (x, y) of image point P of N-vector $\mathbf{m}$ with respect to $\{P_{x\infty}, P_{y\infty}, P_O, P_I\}$:

procedure *projective-coordinates* $(P;\ P_{x\infty},\ P_{y\infty},\ P_O, P_I)$

1. Compute the collineation $\mathbf{A}$ that maps $(\pm\infty, 0)$, $(0, \pm\infty)$, $(0, 0)$, and (f, f) to $P_{x\infty}$, $P_{y\infty}$, P_O, and P_I, respectively, by PROPOSITION 8.
2. Compute $\mathbf{p} = (p_1, p_2, p_3)$ by $\mathbf{p} = \pm N[\mathbf{A}^{\mathrm{T}}\mathbf{m}]$.
3. Return $(x, y) = (fp_1/p_3,\ fp_2/p_3)$.

Using this, we can obtain the following procedure for computing the collineation $\mathbf{A}$ that maps a set of four image points $\{A, B, C, D\}$ in general position to another set of four image points $\{A', B', C', D'\}$ in general position:

procedure *point-collineation* $(A,\ B,\ C,\ D;\ A',\ B',\ D',\ C')$

1. Compute the collineation $\mathbf{A}_1$ that maps $(\pm\infty, 0)$, $(0, \pm\infty)$, $(0, 0)$, and (f, f) to A, B, C, and D, respectively, by PROPOSITION 8.
2. Compute the collineation $\mathbf{A}_2$ that maps $(\pm\infty, 0)$, $(0, \pm\infty)$, $(0, 0)$, and (f, f) to A', B', C', and D', respectively, by PROPOSITION 8.
3. Return collineation $\mathbf{A} = \mathbf{A}_1^{-1}\mathbf{A}_2$.

Since the collineation for image lines is contragradient to the corresponding collineation for image points (Section 2.3), we can also obtain a procedure for computing the collineation that maps a set of four image lines $\{l_1, l_2, l_3, l_4\}$ in general position to another set of four image lines $\{l'_1, l'_2, l'_3, l'_4\}$ in general position by making use of the *polarity* between image points and image lines [5].

If image line l has N-vector $\mathbf{n}$, the image point P having the same N-vector $\mathbf{n}$ is called the *pole* of image line l, which is also called the *polar* of image point P (with respect to the "absolute conic"; see [5] for the details). Thus,

procedure *line-collineation* $(l_1,\ l_2,\ l_3,\ l_4;\ l'_1,\ l'_2,\ l'_3, l'_4)$

1. Compute the collineation $\mathbf{A}$ that maps the poles of l_1, l_2, l_3, and l_4 to the poles of l'_1, l'_2, l'_3, and l'_4, respectively.
2. Return collineation $(\mathbf{A}^{-1})^{\mathrm{T}}$.

3.3. *Harmonic Range of Points*

A set of collinear points is called a *range*, and the line passing through them is called its *axis*. A range of four points $\{A, B, C, D\}$ is called a *harmonic range* if

$$[ABCD] = -1. \tag{40}$$

We also say that points A and B *harmonically divide* points C and D. From Eq. (9), we see that if $\{A, B, C, D\}$ is a harmonic range, then so are $\{A, B, D, C\}$, $\{B, A, C, D\}$, $\{B, A, D, C\}$, $\{C, D, A, B\}$, $\{C, D, B, A\}$, $\{D, C, A, B\}$, and $\{D, C, B, A\}$. The following is a direct consequence of PROPOSITION 2:

PROPOSITION 9. *Let* $\{A, B, C, D\}$ *be a harmonic range, and* $\mathbf{n}$ *the N-vector of its axis. If* $\mathbf{m}_A$, $\mathbf{m}_B$, $\mathbf{m}_C$, *and* $\mathbf{m}_D$ *are their respective N-vectors, the N-vector of one, say* $\mathbf{m}_D$, *is determined from the N-vectors of the rest in the form*

$$\mathbf{m}_D = \pm N[|\mathbf{m}_B, \mathbf{m}_C, \mathbf{n}|\mathbf{m}_A + |\mathbf{m}_A, \mathbf{m}_C, \mathbf{n}|\mathbf{m}_B]. \tag{41}$$

Harmonic ranges of image points play an important role in 3-D interpretation of images as the following proposition suggests (Fig. 6a):

PROPOSITION 10. *Let* $\bar{A}$ *and* $\bar{B}$ *be distinct space points, and let* $\bar{C}$ *be their midpoint. If* A, B, *and* C *are their corresponding image points, and if* P_∞ *is the vanishing point of the space line passing through the space points* $\bar{A}$ *and* $\bar{B}$, *then* $\{P_\infty, C, A, B\}$ *is a harmonic range.*

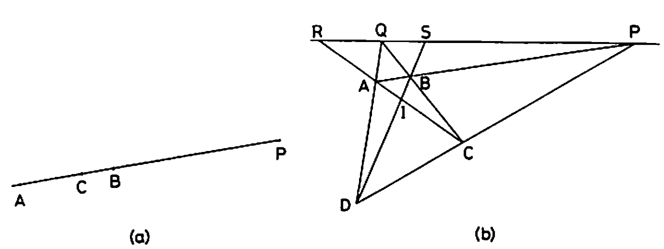


FIG. 6. (a) Harmonic range of image points. (b) Complete quadrangle.

Proof. If $\bar{P}_\infty$ is the ideal point of the space line passing through $\bar{A}$ and $\bar{B}$, then $[\bar{P}_\infty\bar{C}\bar{A}\bar{B}] = -1$ from Eq. (8). The perspective invariance of cross ratio (COROLLARY 3) implies $[P_\infty CAB] = [\bar{P}_\infty\bar{C}\bar{A}\bar{B}] = -1$. ■

Let A, B, C, and D be four coplanar points in general position. Let P be the intersection of AB and CD, and Q the intersection of BC and DA. Such a set of six points $\{A, B, C, D, P, Q\}$ is called a *complete quadrangle* (Fig. 6b). The following is one of the fundamental theorems of projective geometry:

THEOREM 4. *Let $\{A, B, C, D, P, Q\}$ is a complete quadrangle. Let R be the intersection of AC and PQ, and S the intersection of BD and PQ. Let I be the intersection of AC and BD. Then the following are all harmonic ranges:*

$$\{R, I, A, C\}, \quad \{S, I, B, D\}, \quad \{P, Q, R, S\}. \tag{42}$$

Proof. Since any four coplanar points in general position can be mapped to any four coplanar points in general position by some collineation (PROPOSITION 6), the quadrangle $ABCD$ can be mapped to a rectangle (Fig. 7a). Then R and S are mapped to the ideal points of the two diagonals AD and BC, respectively, and $|AI| = |CI|$ and $|BI| = |DI|$. Hence, $\{R, I, A, C\}$ and $\{S, I, B, D\}$ are both harmonic ranges (PROPOSITION 10). The projective invariance of cross ratio (THEOREM 3) implies that $\{R, I, A, C\}$ and $\{S, I, B, D\}$ are both harmonic ranges in any configuration. On the other hand, the quadrangle $ABCD$ can also be mapped to an isosceles trapezoid (Fig. 7b). Then $|RQ| = |SQ|$ and P is the ideal point of line RQS. Hence, $\{P, Q, R, S\}$ is a harmonic range and is so in any configuration due to the projective invariance of cross ratio (THEOREM 3). ■

This theorem itself is well known as a mathematical fact, but if this is combined with our computational procedure of PROPOSITION 9, we can obtain many useful computational techniques. For example, the N-vector of R is computed in terms of the N-vectors of A, C, and I, and the N-vector of S in terms of the N-vectors of B, D, and I (see Fig. 6b) by

$$\mathbf{m}_R = \pm N[|\mathbf{m}_C, \mathbf{m}_I, \mathbf{n}_{AC}|\mathbf{m}_A + |\mathbf{m}_A, \mathbf{m}_I, \mathbf{n}_{AC}|\mathbf{m}_C], \tag{43}$$

$$\mathbf{m}_S = \pm N[|\mathbf{m}_D, \mathbf{m}_I, \mathbf{n}_{BD}|\mathbf{m}_B + |\mathbf{m}_B, \mathbf{m}_I, \mathbf{n}_{BD}|\mathbf{m}_D], \tag{44}$$

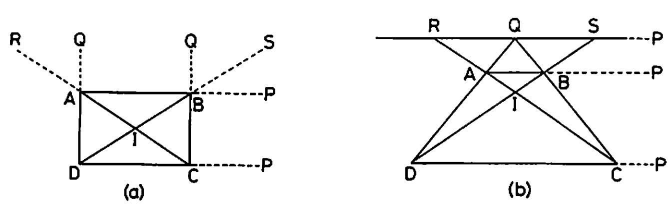


FIG. 7. (a) Rectangle. (b) Isosceles trapezoid.

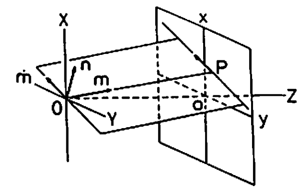


FIG. 8. The N-velocity $\dot{\mathbf{m}}$ of a moving image point and the N-vector n of its trajectory.

where $\mathbf{m}_A$, $\mathbf{m}_B$, etc. are the N-vectors of image points A, B, etc., and $\mathbf{n}_{AC}$ and $\mathbf{n}_{BD}$ are the N-vectors of image lines AC and BD, respectively.

If $ABCD$ is a projection of a quadrilateral (a rectangle or a square in particular) in the scene, image points R and S are the vanishing points of the diagonals, which indicates the 3-D orientations of the corresponding space lines (THEOREM 1), whereas image line RS is the vanishing line of the quadrilateral, which indicates the orientation of the corresponding surface in the scene (THEOREM 2).

4. 3-D MOTION AND FOCUSES OF EXPANSION

4.1. *N-Velocities and Trajectories*

Consider a space point of N-vector $\mathbf{m}$ moving in the scene. Let us call the time derivative $\dot{\mathbf{m}}$ of $\mathbf{m}$ the *N-velocity* of the point [5]. Note that N-velocity $\dot{\mathbf{m}}$ is *not* normalized into a unit vector. Since the N-vector $\mathbf{m}$ is a unit vector, differentiation of $\|\mathbf{m}\|^2 = (\mathbf{m}, \mathbf{m}) = 1$ yields

PROPOSITION 11. *The N-vector and the N-velocity of a moving point are orthogonal to each other:*

$$(\mathbf{m}, \dot{\mathbf{m}}) = 0. \tag{45}$$

Projection of a translating space point defines a straight trajectory on the image plane. Its N-vector is given as follows:

PROPOSITION 12. *If $\mathbf{m}$ and $\dot{\mathbf{m}}$ are the N-vector and the N-velocity, respectively, of a translating space point, the N-vector of its trajectory on the image plane is*

$$\mathbf{n} = \pm N[\mathbf{m} \times \dot{\mathbf{m}}]. \tag{46}$$

Proof. Consider the plane passing through the viewpoint O and intersecting the image plane along the trajectory. Since $\mathbf{m}$ and $\dot{\mathbf{m}}$ are both contained in this plane (Fig. 8a), the unit surface normal to this plane (i.e., the N-vector of the trajectory) is given by Eq. (46). ■

4.2. *Focus of Expansion*

As is well known, projections of translating space points seem to be moving on the image plane away from or toward a fixed point, which is known as the *focus of* *expansion* (Fig. 8b). This fact is obvious from THEOREM 1 and COROLLARY 1, since the focus of expansion is simply the vanishing point of the trajectories in the scene. Thus,

THEOREM 5. *A space point translating in the direction of unit vector* **u** *has, when projected onto the image plane, a focus of expansion whose N-vector is* $\pm\mathbf{u}$.

COROLLARY 4. *Projections of rigidly translating space points have a common focus of expansion on the image plane.*

The focus of expansion is easily computed in terms of N-vectors if projections of multiple space points translating in the scene are observed. The details of the computational procedures are discussed in [5]. Here, we point out that the focus of expansion can also be computed from an image motion of a *single* space point if its projection is observed over at least *three* frames at known times t_1, t_2, and t_3, provided that the space point is translating *with a constant velocity.*

PROPOSITION 13. *If* $\mathbf{m}_\alpha$ *is the N-vector of a space point translating with a constant velocity at time* t_α, $\alpha = 1, 2, 3$, *and if* **n** *is the N-vector of its trajectory, the N-vector* **u** *of the focus of expansion is given by*

$$\mathbf{u} = \pm N\left[|\mathbf{m}_3, \mathbf{m}_2, \mathbf{n}|\mathbf{m}_1 - \frac{t_2 - t_3}{t_2 - t_1}|\mathbf{m}_1, \mathbf{m}_2, \mathbf{n}|\mathbf{m}_3\right], \quad (47)$$

provided that the three image points are all distinct.

Proof. Let $P(t_\alpha)$ be the image points at times t_α, $\alpha = 1, 2, 3$, and P_∞ the focus of expansion (Fig. 9). From the perspective invariance of cross ratio (COROLLARY 3), their cross ratio is given by

$$[P_\infty P(t_2)P(t_1)P(t_3)] = \frac{t_2 - t_3}{t_2 - t_1}. \quad (48)$$

Hence, P_∞ is computed from the N-vectors of $P(t_\alpha)$, $\alpha = 1, 2, 3$, by PROPOSITION 2 in the form of Eq. (47). ∎

In particular, if t_1, t_2, and t_3 are at equal time intervals, the set $\{P_\infty, P(t_2), P(t_1), P(t_3)\}$ is a harmonic range. Hence,

COROLLARY 5. *If* $\mathbf{m}_1$, $\mathbf{m}_2$ and $\mathbf{m}_3$ *are the N-vectors of a space point translating with a constant velocity at times* t_1, t_2, *and* t_3 *in that order at equal time intervals, and if* **n** *is the N-vector of its trajectory, the N-vector* **u** *of the focus of expansion is given by*

$$\mathbf{u} = \pm N[|\mathbf{m}_3, \mathbf{m}_2, \mathbf{n}|\mathbf{m}_1 + |\mathbf{m}_1, \mathbf{m}_2, \mathbf{n}|\mathbf{m}_3], \quad (49)$$

provided that the three image points are all distinct.

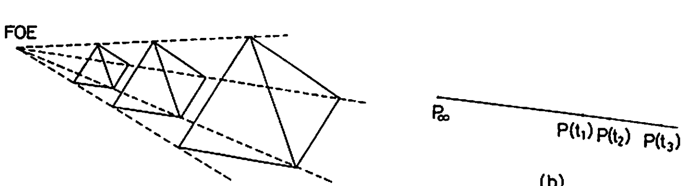


FIG. 9. (a) The focus of expansion. (b) The focus of expansion can be computed from a single point moving over three frames.

Conversely, if distinct projections $P(t_1)$ and $P(t_2)$ of a translating space point at two different times are observed, and if the focus of expansion P_∞ is known, the position of the image point at an arbitrarily specified time t can be located on the trajectory by assuming that the corresponding space point is translating with a constant velocity. All we need to do is define on the trajectory the "coordinate" $[P]$ with respect to $\{P_\infty, P(t_1,), P(t_2)\}$. Then, $[P]$ indicates the time of passage scaled so that $t_1 = 0$ and $t_2 = 1$ (PROPOSITION 4).

The focus of expansion can also be compute from a *single* time instance if the N-vector **m**, the N-velocity $\dot{\mathbf{m}}$, and the *acceleration* $\ddot{\mathbf{m}}$ are given.

PROPOSITION 14. *If* **m** *is the N-vector of a space point translating with a constant velocity, the N-vector* **u** *of the focus of expansion is given by*

$$\mathbf{u} = \pm N[\|\dot{\mathbf{m}}\|^2\dot{\mathbf{m}} - \tfrac{1}{2}(\dot{\mathbf{m}}, \ddot{\mathbf{m}})\mathbf{m}]. \quad (50)$$

Proof. See COROLLARY 5. If we put $\Delta t = t_3 - t_2 = t_2 - t_1$ and drop subscript 1, we can write

$$\mathbf{m}_2 = \mathbf{m} + \dot{\mathbf{m}}\Delta t + \tfrac{1}{2}\ddot{\mathbf{m}}(\Delta t)^2 + O(\Delta t)^3,$$
$$\mathbf{m}_3 = \mathbf{m} + 2\dot{\mathbf{m}}\Delta t + 2\ddot{\mathbf{m}}(\Delta t)^2 + O(\Delta t)^3, \quad (51)$$

where $O(\ldots)^n$ denotes terms of order n or higher in .... Substituting these into Eq. (49), we obtain

$$\mathbf{u} = \pm N[(2|\mathbf{m}, \dot{\mathbf{m}}, \mathbf{n}|\dot{\mathbf{m}} - |\mathbf{m}, \ddot{\mathbf{m}}, \mathbf{n}|\,\mathbf{m})(\Delta t)^2 + O(\Delta t)^2]. \quad (52)$$

The operand of the normalization operator $N[\cdot]$ can be multiplied by any scalar. Dividing the operand by $2(\Delta t)^2$ and taking the limit $\Delta t \to 0$, we obtain

$$\mathbf{u} = \pm N[|\mathbf{m}, \dot{\mathbf{m}}, \mathbf{n}|\dot{\mathbf{m}} - \tfrac{1}{2}|\mathbf{m}, \ddot{\mathbf{m}}, \mathbf{n}|\mathbf{m}]. \quad (53)$$

The N-vector **n** of the trajectory is given by $\mathbf{n} = \pm N[\mathbf{m} \times \dot{\mathbf{m}}]$ (PROPOSITION 11). Since multiplication by a constant does not affect the normalization $N[\cdot]$, the N-vector **n** in Eq. (53) can be replaced by $\mathbf{m} \times \dot{\mathbf{m}}$. If we note that $\|\mathbf{m}\| = 1$ and $(\mathbf{m}, \dot{\mathbf{m}}) = 0$ (PROPOSITION 11), we see that

$$|\mathbf{m}, \dot{\mathbf{m}}, \mathbf{m} \times \dot{\mathbf{m}}| = (\mathbf{m} \times \dot{\mathbf{m}}, \mathbf{m} \times \dot{\mathbf{m}}) = \|\mathbf{m}\|^2\|\dot{\mathbf{m}}\|^2 - (\mathbf{m}, \dot{\mathbf{m}})^2 = \|\dot{\mathbf{m}}\|^2, \quad (54)$$

$$
\begin{aligned}
|\mathbf{m}, \ddot{\mathbf{m}}, \mathbf{m} \times \dot{\mathbf{m}}| &= (\mathbf{m} \times \ddot{\mathbf{m}}, \mathbf{m} \times \dot{\mathbf{m}}) \\
&= (\mathbf{m}, \mathbf{m})(\ddot{\mathbf{m}}, \dot{\mathbf{m}}) - (\mathbf{m}, \dot{\mathbf{m}})(\ddot{\mathbf{m}}, \mathbf{m}) \\
&= (\dot{\mathbf{m}}, \ddot{\mathbf{m}})\mathbf{m}. \qquad (55)
\end{aligned}
$$

If these are substituted into Eq. (53), we obtain Eq. (50). ■

5. CONCLUDING REMARKS

In this paper, we have presented a new computational formulation of cross ratio with a view to applications to computer vision problems by extending the framework of computational projective geometry of Kanatani [5]. Many of the facts shown in this paper are in themselves well known in projective geometry [13]. However, the purpose of this paper is not to show these facts. Rather, our interest is in the computational procedures for them. In this sense, our formulation is *not* orthodox mathematics.

The theme of projective geometry as mathematics is "generality" and "logical consistency", and hence "computational aspects" are not central [13]. To put it differently, projective geometry can be regarded as a mature branch of mathematics for the very reason that it is no longer concerned with the "real" world. In this paper, the perspective invariance of cross ratio (THEOREM 3) was first proved by assuming the assertions in Section 2.3, and then PROPOSITIONS 6 and 7 were derived. This deviates from the orthodox approach. In fact, THEOREM 3 should be proved directly from axioms of projective geometry, and then the assertions in Section 3 should be derived by using PROPOSITIONS 6 and 7, which can be obtained from THEOREM 3.

In such an axiomatic construction, one first considers a mapping between two planes defined in such a way that one plane is projected onto the other from an external "light source." Such a mapping is called a *perspective transformation,* or simply *perspectivity.* A *projective transformation,* or simply *projectivity,* is a mapping obtained by a finite number of compositions of perspectivities. In this paper, we did not take such an axiomatic approach in favor of computational convenience for real image applications (see [13] for the traditional approach).

As typical examples, we constructed procedures for computing the 3-D orientation of a planar shape from its 2-D projection image and the focus of expansion from an image trajectory of a single point by taking advantage of the perspective invariance of cross ratio and projective coordinates, and the resulting 3-D interpretation of harmonic range. The computational definition of cross ratio given in this paper is expected to provide a theoretical foundation to a wide range of computer vision applications where cross ratio is involved.

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