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# NOTE

# Errors of the Incremental Method for Curves

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This is a short note on the numerical errors of the incremental method for curve generation using the forward difference, a basic principle for graphic display, digital plotting, and numerical control. Both the maximum possible errors and the statistical behavior are analyzed.

## 1. INTRODUCTION

Curve generation is one of the most important techniques in computer graphics. In it, polynomials are most frequently used, because they can approximate any smooth curve if their degrees are sufficiently high. In practice, however, polynomials of low degrees, say the third, are chosen and patched together—Hermite curves, B-spline curves, or Bézier curves, for example [1–3]. In computation, the incremental method is of primary importance, because it requires only additions and subtractions and the number of operations at each step equals the degree of the polynomial. Moreover, it also is suitable for hardware implementation in the form of the digital differential analyzer (DDA), and the same technique applies to other problems as well—plotters, numerical control, etc. Let $f(k)$, $k = 1, 2, 3, \ldots$, be a polynomial of degree $n$. Then the (forward) difference $\Delta f(k) = f(k+1) - f(k)$ is a polynomial of degree $n-1$, the second differene $\Delta^2 f(k) = \Delta f(k+1) - \Delta f(k)$ is a polynomial of degree $n-2, \ldots$, and the $n$th difference $\Delta^n f(k) = \Delta^{n-1} f(k+1) - \Delta^{n-1} f(k)$ is a constant, say $C$. Hence if we first compute the initial values $f(0), \Delta f(0), \ldots, \Delta^n f(0)$ $(= C)$, we can successively compute $f(k)$, using only additions and subtractions, by $f(k+1) = f(k) + \Delta f(k)$, $\Delta f(k+1) = \Delta f(k) + \Delta^2 f(k), \ldots, \Delta^{n-1} f(k+1) = \Delta^{n-1} f(k) + C$. This is the well known incremental method [1, 2].

One of the problems with the incremental method is the propagation of the error. The rounding or truncation errors at each step are no problem, because the operations are additions and subtractions. They can be computed exactly in most cases, and even in the worst case the errors grow at most linearly in $k$. However, even if the addition/subtraction is exact, the error grows drastically if the initial values have only a slight amount of error. In this note, we analyze the propagation of the initial errors. We consider both the worst case and the statistical behavior, but we will see that the worst case is very likely to occur. However, the knowledge obtained here can be used to predict how far we can proceed within a given error bound, which, for example, is determined by the screen resolution. Hence, we can tell at which step we must recompute the function values directly in order to clear the errors accumulated so far.



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## 2. THE MAXIMUM ERRORS

Suppose $f(k)$ is given by $f(k) = F(a + hk)$, where $F(x)$ is a real-valued polynomial and $a$ and $h$ are real-valued constants. In order to compute $f(0), \Delta f(0), \ldots, \Delta^n f(0)$ by the definition of the forward difference, it is necessary to compute $f(0), f(1), f(2), \ldots, f(n)$. Then, the direct computation of the differences according to the definition is very efficient, and requires only $n(n+1)/2$ additions/subtractions. Since the computation involves real values, the computed values have rounding or truncation errors. Let $\varepsilon_i$ be the error involved in $f(i)$, and assume that the addition/subtraction thereafter can be executed without errors. (Later, it is easily seen that the neglected effect is really negligible.) If we compute the differences according to the definition under this assumption, we are actually computing $\Delta^l f(0) = \sum_{m=0}^{l}(-1)^{l-m}\binom{l}{m} f(m)$, as is well known [4], and hence the error $\eta_l$ involved in $\Delta^l f(0)$ is

$$\eta_l = \sum_{m=0}^{l} (-1)^{l-m}\binom{l}{m}\varepsilon_m. \tag{1}$$

Hence, if the maximum absolute value of the initial errors $\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_n$ is $\varepsilon$, we obtain the following error bound:

$$|\eta_l| \le \sum_{m=0}^{l}\binom{l}{m}|\varepsilon_m| \le \varepsilon \sum_{m=0}^{l}\binom{l}{m} = 2^l\varepsilon. \tag{2}$$

This bound cannot be improved, because the equalities hold for $\varepsilon_0 = \varepsilon$, $\varepsilon_1 = -\varepsilon$, $\varepsilon_2 = \varepsilon, \ldots$ .

Now, we proceed to the error propagation of the incremental method. According to our assumption, if the values of $f(0), f(1), \ldots, f(n)$ are once computed up to some errors, the following computations of differences and additions of increments are performed without errors. This means that we are computing nothing but the exact extrapolation of the initial values. Hence, using the *Newton forward interpolation formula*, we can express the computed value of $f(k)$ as $f(k) = \sum_{l=0}^{n} k^{[l]}\Delta^l f(0)/l!$, where $k^{[l]} = k(k-1)(k-2)\cdots(k-l+1)$ [4]. The error $\varepsilon_k$ involved in $f(k)$ is

$$\varepsilon_k = \sum_{l=0}^{n} \frac{k^{[l]}}{l!}\eta_l. \tag{3}$$

Thus, we obtain the following error bound:

$$\frac{|\varepsilon_k|}{\varepsilon} \le \sum_{l=0}^{n} \frac{k^{[l]}}{l!}\frac{|\eta_l|}{\varepsilon} \le \sum_{l=0}^{n} \frac{2^l}{l!}k^{[l]} = \frac{2^n}{n!}k^n + O(k^{n-1}). \tag{4}$$

## 3. STATISTICAL BEHAVIOR OF ERRORS

The error bounds obtained in the previous section describe the maximum possible errors, but such errors might not result due to mutual cancellation. Statistical treatment is available to check if that is the case [4]. Let the errors $\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_n$ of the computed values $f(0), f(1), \ldots, f(n)$ be regarded as mutually independent random variables with mean 0 and variance $\sigma^2$. For example, if they are uniformly distributed in the interval $[-\varepsilon, \varepsilon]$, then $\sigma^2 = \varepsilon^2/3$. Then, in view of Eq. (1), the mean, the variance, and the covariance of $\eta_0, \eta_1, \ldots, \eta_n$ become (after some manipulation involving binomial coefficients)

$$E[\eta_l] = \sum_{m=0}^{l} (-1)^{l-m} \binom{l}{m} E[\varepsilon_m] = 0, \tag{5}$$

$$V[\eta_l] = \sum_{m=0}^{l} \binom{l}{m}^2 V[\varepsilon_m] = \sigma^2 \binom{2l}{l}, \tag{6}$$

$$C[\eta_l, \eta_{l'}] = E[\eta_l \eta_{l'}] = (-1)^{l+l'} \sigma^2 \binom{l+l'}{l}, \tag{7}$$

where $E$, $V$, and $C$ designate the mean, the variance, and the covariance, respectively.

Then, from Eq. (2), we can calculate the mean and the variance of the errors involved in $f(k)$ at each step as follows:

$$E[\varepsilon_k] = \sum_{l=0}^{n} \frac{k^{[l]}}{l!} E[\eta_l] = 0, \tag{8}$$

$$\begin{aligned} V[\varepsilon_k] &= \sum_{l=0}^{n} \sum_{l'=0}^{n} \frac{k^{[l]} k^{[l']}}{l! l'!} E[\eta_l \eta_{l'}] \\ &= \sigma^2 \sum_{l=0}^{n} \sum_{l'=0}^{n} \frac{(-1)^{l+l'} (l+l')!}{(l!)^2 (l'!)^2} k^{[l]} k^{[l']} \\ &= \sigma^2 \left[ \frac{(2n)!}{(n!)^4} k^{2n} + O(k^{2n-1}) \right]. \end{aligned} \tag{9}$$

Since $1/2n \leq (2n)!/2^{2n}(n!)^2 \leq 1/2$, we can see that if $k$ is sufficiently large

$$\sqrt{V[\varepsilon_k]}/\sigma \sim \alpha 2^n k^n / n!, \tag{10}$$

where $1/\sqrt{2n} \leq \alpha \leq 1/\sqrt{2}$.

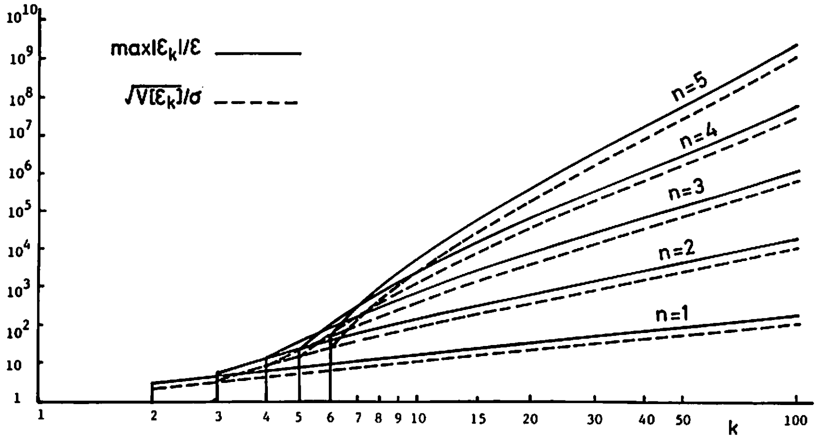


FIG. 1. The maximum and the standard deviation of the error.

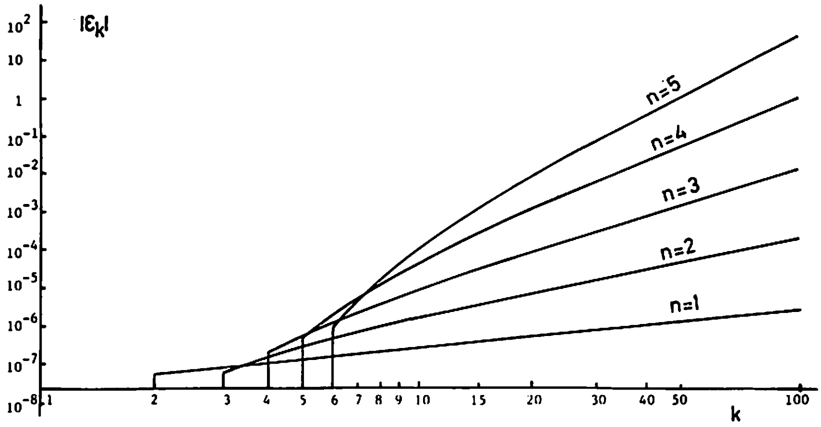


FIG. 2. Errors of numerical experiments.

## 4. CONCLUSION

Comparing this estimate with Eq. (4), we can see that the worst case of Eq. (4) is very likely to occur. Figure 1 plots Eqs. (4) and (9), and Fig. 2 shows typical results of a numerical experiment. These figures support our assertion. On the other hand, we can predict the error at each step from our result. Hence, if an acceptable error bound is given, by the screen resolution for example, we can forsee at which step we must stop the process and renew the computation by resorting to the definition of the desired curve. This prediction depends only on the number of significant digits in the computation and not on what the curve is. For example, if $M$ is a given acceptable maximum error magnification ratio, i.e., $|\varepsilon_k|/\varepsilon \leq M$, we can use the approximation $M \sim 2^n k^n/n!$ to obtain

$$0 \leq k \leq \sqrt[n]{n!M}/2. \tag{11}$$

As we have seen, the incremental method is very susceptible to initial errors, and the allowable range (11) of $k$ is fairly small. If the degree $n$ of the polynomial is large, the maximum allowable $k$ becomes by application of the Stirling formula $n! \sim \sqrt{2\pi n}\, n^n/e^n$

$$k_{\max} \sim \sqrt[2n]{2\pi n}\, n \sqrt[n]{M}/2e. \tag{12}$$

If $k_{\max}$ is given, on the other hand, Eq. (11) or (12) gives the error magnification ratio $M$, which determines the degree of precision necessary for the computation of initial values.

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