Closed-Form Expression for Focal Lengths from the Fundamental Matrix

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ABSTRACT

We describe an algorithm for decomposing a fundamental matrix computed from point correspondences over two images into the focal lengths of the two images and the camera motion parameters in a closed-form expression. Our algorithm is based on the decomposability condition of the essential matrix expressed in terms of its scalar invariants. We give a complete analysis for degenerate camera configurations. We also describe an algorithm for computing a single focal length in the degenerate case and analyze the indeterminacy condition.

Keywords: structure from motion, fundamental matrix, self-calibration, focal length, invariant, polynomial system, degenerate configuration.

1. INTRODUCTION

In reconstructing the 3-D structure from point correspondences over two images taken by two uncalibrated cameras, all available information is encoded in the *fundamental matrix* $\boldsymbol{F}$ if no prior knowledge is available about the scene [3, 18]. Since $\boldsymbol{F}$ is a singular matrix of rank 2 defined up to scale, it has seven degrees of freedom. The relative motion of the two cameras are specified by a translation vector $\boldsymbol{t}$ and a rotation matrix $\boldsymbol{R}$. Since the absolute scale of the translation is indeterminate, the motion parameters $\{\boldsymbol{t},\ \boldsymbol{R}\}$ have five degrees of freedom. It follows that only up to two camera parameters can be recovered if the camera motion is arbitrary.

A practical choice for the two parameters is the *focal lengths* f and f' of the two cameras, since other parameters can be pre-calibrated and fixed while zooming usually changes freely as the camera moves. Strictly speaking, the *principal point* (the intersection of the optical axis with the image plane) may slightly move as zooming changes, but regarding it as a fixed point is known to be a good approximation. Once we have obtained the focal lengths f and f' and the motion parameters $\{\boldsymbol{t},\ \boldsymbol{R}\}$, the 3-D structure of the scene can be reconstructed up to scale by triangulation [9, 10].

Hartley [5] presented an analytic procedure for computing the focal lengths f and f' from the fundamental matrix $\boldsymbol{F}$. The solution is obtained by applying the singular value decomposition (SVD) and solving linear equations in four unknowns. Pan et al. [16, 17] reduced this problem to solving cubic equations. Newsam et al. [15] refined these algorithms into a combination of SVD and linear equations in three unknowns. They also derived the degeneracy condition for the solution to be indeterminate. Bougnoux [1] presented a closed-form formula for f in terms of the fundamental matrix $\boldsymbol{F}$ and the *epipoles* $\boldsymbol{e}$ and $\boldsymbol{e}'$ (eigenvectors of $\boldsymbol{F}^\top$ and $\boldsymbol{F}$, respectively, for eigenvalue 0) (see Appendix A). In this paper, we present a closed-form expression for the focal lengths f and f *in terms of the elements of the fundamental matrix* $\boldsymbol{F}$ *alone*: it does not involve SVD, linear equations, or epipoles.

The significance of this paper is mostly theoretical, since all the algorithms produce exactly the same solution for the same input. The intuition that leads to our algorithm is *group-theoretical invariance* [8]: if an algorithm for computing f and f' from $\boldsymbol{F}$ is to exist, *the choice of the image coordinate system in each frame should not affect the structure of the algorithm.* In particular, an arbitrary image coordinate rotation around the principal point in each frame should not affect the solution. This means that f and f' should be expressed in terms of *scalar invariants* of $\boldsymbol{F}$, since f and f' are themselves scalar invariants with respect to image coordinate rotations.

We first describe the algorithm and then give a justification for it. Next, we give a complete analysis for degenerate configurations in which the solution is indeterminate. Our result completely agrees with that of Newsam et al. [15]. Finally, we describe an algorithm for computing a single focal length in the degenerate case. We show that the solution is indeterminate only for the singular configurations that Brooks et al. [2] found for a horizontally-constrained stereo head.

2. DESCRIPTION OF THE ALGORITHM

The inner product of vectors $\boldsymbol{a} = (a_i)$ and $\boldsymbol{b} = (b_i)$ is denoted by $(\boldsymbol{a}, \boldsymbol{b}) = \sum_{i=1}^3 a_i b_i$. The norm of a vector $\boldsymbol{a} = (a_i)$ is denoted by $\|\boldsymbol{a}\| = \sqrt{\sum_{i=1}^3 a_i{}^2}$. The norm of a matrix $\boldsymbol{M} = (M_{ij})$ is defined by $\|\boldsymbol{M}\| = \sqrt{\sum_{i,j=1}^3 M_{ij}{}^2}$. We let $\boldsymbol{k} = (0,0,1)^\top$ throughout this paper.

We assume that the principal point in each image is known and take it to be the image coordinate origin. We also assume that the aspect ratio and the skewness of the image frame is known and define image coordinate systems in such a way that the effective aspect ratio is 1 with no skew. The *fundamental matrix* $\boldsymbol{F}$ is a matrix of rank 2 such that the *epipolar equation*

$$(\begin{pmatrix} x \\ y \\ 1 \end{pmatrix}, \boldsymbol{F}\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix}) = 0 \tag{1}$$

is satisfied for any point (x, y) in the first image and the corresponding point (x', y') in the second image. If the principal point is not at the coordinate origin but at (u_0, v_0), we first transform $\boldsymbol{F}$ in the form

$$\boldsymbol{F} \leftarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ u_0 & v_0 & 1 \end{pmatrix} \boldsymbol{F} \begin{pmatrix} 1 & 0 & u_0 \\ 0 & 1 & v_0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{2}$$

The algorithm for computing the focal lengths f and f' is described as follows:

Input: A fundamental matrix $\boldsymbol{F}$.

Output: Two focal lengths f and f'.

Procedure:

1. Compute the following quantities:

$$a = \frac{\|\boldsymbol{F}\boldsymbol{F}^\top\boldsymbol{k}\|^2}{\|\boldsymbol{F}^\top\boldsymbol{k}\|^2}, \qquad b = \frac{\|\boldsymbol{F}^\top\boldsymbol{F}\boldsymbol{k}\|^2}{\|\boldsymbol{F}\boldsymbol{k}\|^2},$$

$$c = \frac{(\boldsymbol{k}, \boldsymbol{F}\boldsymbol{k})^2}{\|\boldsymbol{F}^\top\boldsymbol{k}\|^2\|\boldsymbol{F}\boldsymbol{k}\|^2}, \qquad d = \frac{(\boldsymbol{k}, \boldsymbol{F}\boldsymbol{F}^\top\boldsymbol{F}\boldsymbol{k})}{(\boldsymbol{k}, \boldsymbol{F}\boldsymbol{k})}, \quad (3)$$

$$A = \frac{1}{c} + a - 2d, \qquad B = \frac{1}{c} + b - 2d, \quad (4)$$

$$P = 2\Big(\frac{1}{c} - 2d + \frac{1}{2}\|\boldsymbol{F}\|^2\Big),$$

$$Q = -\frac{A+B}{c} + \frac{1}{2}\Big(\|\boldsymbol{F}\boldsymbol{F}^\top\|^2 - \frac{1}{2}\|\boldsymbol{F}\|^4\Big). \quad (5)$$

2. Solve the following quadratic equation in Z:

$$(1 + cP)Z^2 - (cP^2 + 2P + 4cQ)Z + P^2 + 4cPQ + 12AB = 0. \quad (6)$$

3. Choose from among the two solutions the one for which the following is smaller (ideally zero):

$$\Big|Z^3 - 3PZ^2 + 2(P^2 + 2Q)Z - 4\Big(PQ + \frac{4AB}{c}\Big)\Big|. \quad (7)$$

4. Compute

$$X = -\frac{1}{c}\Big(1 + \frac{2B}{Z-P}\Big), \quad Y = -\frac{1}{c}\Big(1 + \frac{2A}{Z-P}\Big). \quad (8)$$

5. Return the following focal lengths f and f':

$$f = \frac{1}{\sqrt{1 + X/\|\boldsymbol{F}^\top\boldsymbol{k}\|^2}}, \quad f' = \frac{1}{\sqrt{1 + Y/\|\boldsymbol{F}\boldsymbol{k}\|^2}}. \quad (9)$$

Since the solution of the quadratic equation (6) can be explicitly expressed by the quadratic formula, the focal lengths f and f' are given as closed-from expressions in $\|\boldsymbol{F}\|^2$, $\|\boldsymbol{F}\boldsymbol{F}^\top\|^2$, and the scalar invariants defined by eqs. (3).

The case in which the above algorithm fails is when (i) $\|\boldsymbol{F}^\top\boldsymbol{k}\|$, $\|\boldsymbol{F}\boldsymbol{k}\|$, or $(\boldsymbol{k}, \boldsymbol{F}\boldsymbol{k})$ vanishes or (ii) $Z = P$. We will later investigate when and how this occurs and show that this corresponds to degenerate camera configurations.

3. COMPUTATION OF THE MOTION

Once f and f' are determined, the motion parameters $\{\boldsymbol{t}, \boldsymbol{R}\}$ are determined by the following procedure [9, 10]:

1. Compute the following *essential matrix*:

$$\boldsymbol{E} = \mathrm{diag}(1, 1, \frac{1}{f})\boldsymbol{F}\mathrm{diag}(1, 1, \frac{1}{f'}). \quad (10)$$

2. Compute the unit eigenvector $\boldsymbol{t}$ of $\boldsymbol{E}\boldsymbol{E}^\top$ for the smallest eigenvalue.
3. Apply SVD to $-\boldsymbol{t} \times \boldsymbol{E}$ as follows.

$$-\boldsymbol{t} \times \boldsymbol{E} = \boldsymbol{V}\boldsymbol{\Lambda}\boldsymbol{U}^\top. \quad (11)$$

4. Compute the rotation $\boldsymbol{R}$ as follows:

$$\boldsymbol{R} = \boldsymbol{V}\mathrm{diag}(1, 1, \det \boldsymbol{V}\boldsymbol{U}^\top)\boldsymbol{U}^\top. \quad (12)$$

In eq. (10), diag$(\cdots)$ denotes the diagonal matrix with diagonal elements $\cdots$ in that order. The left-hand side of eq. (11) is an exterior product of $-\boldsymbol{t}$ and $\boldsymbol{E}$: for a vector $\boldsymbol{a}$ and a matrix $\boldsymbol{A}$, we define $\boldsymbol{a} \times \boldsymbol{A}$ to be the matrix consisting of columns that are the vector products of $\boldsymbol{a}$ and the individual columns of $\boldsymbol{A}$. On the right-hand side of eq. (11), $\boldsymbol{\Lambda}$ is a diagonal matrix with diagonal elements (*singular values*) in non-increasing order; $\boldsymbol{V}$ and $\boldsymbol{U}$ are orthogonal matrices.

However, *the solution is not unique.* Since the fundamental matrix $\boldsymbol{F}$ is determined only up to scale, its sign is indeterminate. Also, the eigenvector $\boldsymbol{t}$ of $\boldsymbol{E}\boldsymbol{E}^\top$ can be determined only up to sign. Hence, there exist *four* solutions depending on the choice of the signs of $\pm\boldsymbol{E}$ and $\pm\boldsymbol{t}$. If $\{\boldsymbol{t}, \boldsymbol{R}\}$ are the true solution, the other solutions are $\{\boldsymbol{t}, \boldsymbol{R}\}$, $\{\boldsymbol{t}, \boldsymbol{I}_{\boldsymbol{t}}\boldsymbol{R}\}$, $\{-\boldsymbol{t}, \boldsymbol{I}_{\boldsymbol{t}}\boldsymbol{R}\}$, and $\{-\boldsymbol{t}, \boldsymbol{R}\}$, where $\boldsymbol{I}_{\boldsymbol{t}} = 2\boldsymbol{t}\boldsymbol{t}^\top - \boldsymbol{I}$ denotes the 180° rotation around $\boldsymbol{t}$ [9]. This indeterminacy stems from the fact that the mathematical expression for perspective projection is the same if the object is *behind* the camera. The four solution correspond to the pairs {front, front}, {front, behind}, {behind, front}, {behind, behind} for the relative positions of the object to the two cameras. In 3-D reconstruction, the one for which the object is in front of the two cameras is chosen [10].

The use of SVD in the above procedure is merely a convenience of description; the translation $\boldsymbol{t}$ and the rotation $\boldsymbol{R}$ can be obtained by direct manipulations without SVD [12]. The advantage of the above procedure is the fact that it implicitly incorporates least-squares optimization so that it produces for an arbitrary matrix $\boldsymbol{E}$ an exact unit vector $\boldsymbol{t}$ and an exact rotation matrix $\boldsymbol{R}$ such that $\boldsymbol{t} \times \boldsymbol{R}$ is approximately a multiple of $\boldsymbol{E}$ by a constant.

4. JUSTIFICATION OF THE ALGORITHM

If the fundamental matrix $\boldsymbol{F}$ is defined by focal lengths f and f' and motion parameters $\{\boldsymbol{t}, \boldsymbol{R}\}$, the matrix $\boldsymbol{E}$ (the *essential matrix*) defined by eq. (10) should have the expression $\boldsymbol{t} \times \boldsymbol{R}$ [9, 10]. A matrix $\boldsymbol{E}$ has the form $\boldsymbol{t} \times \boldsymbol{R}$ for a vector $\boldsymbol{t}$ and a rotation matrix $\boldsymbol{R}$ if and only if one of the singular values of $\boldsymbol{E}$ is zero and the other two are equal (the *decomposability condition* [7]). *This statement is invariant to coordinate rotations in each frame*, so it can be described in terms of *scalar invariants* of $\boldsymbol{E}$ [8]. In fact, we obtain the following expression [3, 9, 13]:

$$\|\boldsymbol{E}\boldsymbol{E}^\top\|^2 = \frac{1}{2}\|\boldsymbol{E}\|^4. \quad (13)$$

Thus, the focal lengths f and f' are determined by letting

$$K(f, f') = \|\boldsymbol{E}\boldsymbol{E}^\top\|^2 - \frac{1}{2}\|\boldsymbol{E}\|^4, \quad (14)$$

and solving $K(f, f') = 0$, which is a polynomial equation of degree 8 in $1/f$ and $1/f'$ (degree 4 in each). It appears that a single equation is unable to determine two unknowns f and f', but it turns out that the solution is a *singularity* of $K(f, f')$: we have

$$K(f, f') = K_f(f, f') = K_{f'}(f, f') = 0, \quad (15)$$

where the subscript means partial differentiation [10, 13] (see Appendix B).

It can be shown by a lengthy calculation that if we define new variables

$$x = \left(\frac{1}{f}\right)^2 - 1, \qquad y = \left(\frac{1}{f'}\right)^2 - 1, \tag{16}$$

the function K is a polynomial of degree 4 in x and y (degree 2 in each) in the following form:

$$\begin{aligned} K = & (\boldsymbol{k}, \boldsymbol{F}\boldsymbol{k})^4 x^2 y^2 + 2(\boldsymbol{k}, \boldsymbol{F}\boldsymbol{k})^2 \|\boldsymbol{F}^\top \boldsymbol{k}\|^2 x^2 y \\ & +2(\boldsymbol{k}, \boldsymbol{F}\boldsymbol{k})^2 \|\boldsymbol{F}\boldsymbol{k}\|^2 x y^2 + \|\boldsymbol{F}^\top \boldsymbol{k}\|^4 x^2 + \|\boldsymbol{F}\boldsymbol{k}\|^4 y^2 \\ & +4(\boldsymbol{k}, \boldsymbol{F}\boldsymbol{k})(\boldsymbol{k}, \boldsymbol{F}\boldsymbol{F}^\top \boldsymbol{F}\boldsymbol{k}) x y + 2\|\boldsymbol{F}\boldsymbol{F}^\top \boldsymbol{k}\|^2 x \\ & +2\|\boldsymbol{F}^\top \boldsymbol{F}\boldsymbol{k}\|^2 y + \|\boldsymbol{F}\boldsymbol{F}^\top\|^2 - \frac{1}{2}\Big((\boldsymbol{k}, \boldsymbol{F}\boldsymbol{k})^2 x y \\ & +\|\boldsymbol{F}^\top \boldsymbol{k}\|^2 x + \|\boldsymbol{F}\boldsymbol{k}\|^2 y + \|\boldsymbol{F}\|^2\Big)^2. \end{aligned} \tag{17}$$

Assuming that $\|\boldsymbol{F}^\top \boldsymbol{k}\|$, $\|\boldsymbol{F}\boldsymbol{k}\|$, and $(\boldsymbol{k}, \boldsymbol{F}\boldsymbol{k})$ do not vanish, we switch to new variables

$$X = \|\boldsymbol{F}^\top \boldsymbol{k}\|^2 x, \qquad Y = \|\boldsymbol{F}\boldsymbol{k}\|^2 y, \tag{18}$$

and put

$$Z = cXY + X + Y. \tag{19}$$

Then, eq. (17) is written in the form

$$K = 2\Big(g(Z) + AX + BY + \frac{A+B}{c}\Big), \tag{20}$$

where $g(Z)$ is the following quadratic polynomial:

$$g(Z) = \frac{1}{4}Z^2 - \frac{1}{2}PZ + Q. \tag{21}$$

In the above equations, c, A, B, P, and Q are constants defined by eqs. (3), (4), and (5). From eq. (20), the condition $K = K_X = K_Y = 0$ is written as the following three equations:

$$g(Z) + A\Big(X + \frac{1}{c}\Big) + B\Big(Y + \frac{1}{c}\Big) = 0, \tag{22}$$

$$X + \frac{1}{c} = -\frac{2B}{c(Z-P)}, \qquad Y + \frac{1}{c} = -\frac{2A}{c(Z-P)}. \tag{23}$$

Substituting eqs. (23) into eq. (22), we obtain the following cubic equation in Z:

$$Z^3 - 3PZ^2 + 2(P^2+2Q)Z - 4\Big(PQ + \frac{4AB}{c}\Big) = 0. \tag{24}$$

If, on the other hand, eqs. (23) are substituted into eq. (19), we obtain the following cubic equation in Z:

$$Z^3 - \Big(2P - \frac{1}{c}\Big)Z^2 + P\Big(P - \frac{2}{c}\Big)Z + \frac{P^2 - 4AB}{c} = 0. \tag{25}$$

Eliminating Z^3 from eqs. (24) and (25), we obtain the quadratic equation (6). From among the two solutions, we choose the one that satisfies eqs. (24) (or equivalently eq. (25)). Once Z is determined, eqs. (23) determine X and Y, and eqs. (16) and (18) determine the focal lengths f and f'.

5. DEGENERATE CONFIGURATIONS

We translate and scale the xy coordinate system so that the solution comes to the origin $(0,0)$. This is done by using new coordinates (x', y') given by

$$x' = \bar{f}^2(x+1) - 1, \qquad y' = \bar{f}'^2(y+1) - 1, \tag{26}$$

where $\bar{f}$ and $\bar{f}'$ are the true values of f and f'. In terms of these new coordinates, the polynomial K is written in the form

$$\begin{aligned} K = & (\boldsymbol{k}, \bar{\boldsymbol{E}}\boldsymbol{k})^4 x'^2 y'^2 + 2(\boldsymbol{k}, \bar{\boldsymbol{E}}\boldsymbol{k})^2 \|\bar{\boldsymbol{E}}^\top \boldsymbol{k}\|^2 x'^2 y' \\ & +2(\boldsymbol{k}, \bar{\boldsymbol{E}}\boldsymbol{k})^2 \|\bar{\boldsymbol{E}}\boldsymbol{k}\|^2 x' y'^2 + \|\bar{\boldsymbol{E}}^\top \boldsymbol{k}\|^4 x'^2 \\ & +\|\bar{\boldsymbol{E}}\boldsymbol{k}\|^4 y'^2 + 4(\boldsymbol{k}, \bar{\boldsymbol{E}}\boldsymbol{k})(\boldsymbol{k}, \bar{\boldsymbol{E}}\bar{\boldsymbol{E}}^\top \bar{\boldsymbol{E}}\boldsymbol{k}) x' y' \\ & +2\|\bar{\boldsymbol{E}}\bar{\boldsymbol{E}}^\top \boldsymbol{k}\|^2 x' + 2\|\bar{\boldsymbol{E}}^\top \bar{\boldsymbol{E}}\boldsymbol{k}\|^2 y' + \|\bar{\boldsymbol{E}}\bar{\boldsymbol{E}}^\top\|^2 \\ & -\frac{1}{2}\Big((\boldsymbol{k}, \bar{\boldsymbol{E}}\boldsymbol{k})^2 x' y' + \|\bar{\boldsymbol{E}}^\top \boldsymbol{k}\|^2 x' + \|\bar{\boldsymbol{E}}\boldsymbol{k}\|^2 y' \\ & +\|\bar{\boldsymbol{E}}\|^2\Big)^2, \end{aligned} \tag{27}$$

where

$$\bar{\boldsymbol{E}} = \mathrm{diag}(1, 1, \frac{1}{\bar{f}})\boldsymbol{F}\mathrm{diag}(1, 1, \frac{1}{\bar{f}'}). \tag{28}$$

Since $\bar{\boldsymbol{E}}$ is the true value of the essential matrix $\boldsymbol{E}$, it has the form

$$\bar{\boldsymbol{E}} = \boldsymbol{t} \times \boldsymbol{R} \tag{29}$$

for some vector $\boldsymbol{t}$ and some rotation matrix $\boldsymbol{R}$. For this expression, it is easy to show [9, 10] that

$$\|\bar{\boldsymbol{E}}\bar{\boldsymbol{E}}^\top\|^2 = \frac{1}{2}\|\bar{\boldsymbol{E}}\|^4 = 2\|\boldsymbol{t}\|^4. \tag{30}$$

It is also easy to show the following:

$$\begin{gathered} \|\bar{\boldsymbol{E}}^\top \boldsymbol{k}\| = \|\boldsymbol{t}\| \sin\theta, \qquad \|\bar{\boldsymbol{E}}\boldsymbol{k}\| = \|\boldsymbol{t}\| \sin\theta', \\ \|\bar{\boldsymbol{E}}\bar{\boldsymbol{E}}^\top \boldsymbol{k}\| = \|\boldsymbol{t}\|^2 \sin\theta, \qquad \|\bar{\boldsymbol{E}}^\top \bar{\boldsymbol{E}}\boldsymbol{k}\| = \|\boldsymbol{t}\|^2 \sin\theta', \\ (\boldsymbol{k}, \bar{\boldsymbol{E}}\boldsymbol{k}) = -\|\boldsymbol{t}\| \sin\phi \sin\theta \sin\theta', \\ (\boldsymbol{k}, \bar{\boldsymbol{E}}\bar{\boldsymbol{E}}^\top \bar{\boldsymbol{E}}\boldsymbol{k}) = -\|\boldsymbol{t}\|^3 \sin\phi \sin\theta \sin\theta'. \end{gathered} \tag{31}$$

Here, θ is the angle between the translation $\boldsymbol{t}$ and the optical axis direction $\boldsymbol{k}$ of the first camera; θ' is the angle between the translation $\boldsymbol{t}$ and the optical axis direction $\boldsymbol{k}' = \boldsymbol{R}\boldsymbol{k}$ of the second camera; ϕ is the angle between the plane defined by $\boldsymbol{k}$ and $\boldsymbol{t}$ and the plane defined by $\boldsymbol{k}'$ and $\boldsymbol{t}$. Substituting the above expressions into eq. (27), we can easily confirm that $K = K_{x'} = K_{y'} = 0$ at $x' = y' = 0$. The second derivatives at $x' = y' = 0$ are given by

$$\begin{aligned} K_{xx} &= \|\boldsymbol{t}\|^4 \sin^4\theta, \\ K_{xy} &= \|\boldsymbol{t}\|^4 (2\sin^2\phi - 1) \sin^2\theta \sin^2\theta', \\ K_{yy} &= \|\boldsymbol{t}\|^4 \sin^4\theta'. \end{aligned} \tag{32}$$

The necessary condition for indeterminacy of the solution is the vanishing of the determinant $K_{xx}K_{yy} - {K_{xy}}^2$ of the Hessian of $K(x, y)$. We have

$$K_{xx}K_{yy} - K_{xy}^2 = \|\boldsymbol{t}\|^8 \Big(1 - (2\sin^2\phi - 1)^2\Big) \sin^4\theta \sin^4\theta'. \tag{33}$$

This vanishes when $\sin\theta = 0$, or $\sin\theta' = 0$, or $(2\sin^2\phi - 1)^2 = 1$. The last case occurs when $\sin\phi = 0$ or $\sin\phi = \pm 1$. Let us investigate each case separately:

Case 1 **($\sin\theta = 0$):** This occurs when $\theta = 0$ or π, i.e., the optical axis of the first camera is parallel to the translation direction. In this case, we have

$$K(x', y') = \frac{1}{2}\|\boldsymbol{t}\|^4 y'^2 \sin^4\theta'. \tag{34}$$

Evidently, x' is indeterminate.

Case 2 ($\sin\theta' = 0$)**:** This occurs when $\theta' = 0$ or π, i.e., the optical axis of the second camera is parallel to the translation direction. In this case, we have

$$K(x', y') = \frac{1}{2}\|\boldsymbol{t}\|^4 x'^2 \sin^4\theta. \tag{35}$$

Evidently, y' is indeterminate.

Case 3 ($\sin\phi = 0$)**:** This occurs when $\phi = 0$ or π, i.e., the optical axes of the two cameras are coplanar. In this case, we have

$$K(x', y') = \frac{1}{2}\|\boldsymbol{t}\|^4 (x' \sin^2\theta - y' \sin^2\theta')^2. \tag{36}$$

Evidently, we have infinitely many solutions such that $x' \sin^2\theta = y' \sin^2\theta'$.

Case 4 ($\sin\phi = \pm 1$)**:** This occurs $\phi = \pi/2$ or $3\pi/2$, i.e., the plane defined by the optical axis of the first camera and $\boldsymbol{t}$ and the plane define by the optical axis of the second camera and $\boldsymbol{t}$ are orthogonal. In this case, we have

$$K(x', y') = \frac{1}{8}\|\boldsymbol{t}\|^4 (x'y' \sin^2\theta \sin^2\theta' + x' \sin^2\theta + y' \sin^2\theta')^2. \tag{37}$$

Evidently, we have infinitely many solutions such that $xy \sin^2\theta \sin^2\theta' + x \sin^2\theta + y \sin^2\theta' = 0$.

We thus obtain the following proposition, which is identical to that given by Newsam et al. [15]:

Proposition 1 *The focal lengths are indeterminate if and only if (i) the optical axes of the two cameras are coplanar or (ii) the plane defined by the optical axis of the first camera and the translation direction and the plane define by the optical axis of the second camera and the translation direction are orthogonal.*

We have also observed the following fact:

Proposition 2 *The solution is completely indeterminate if the two optical axes and the translation direction are collinear. In other degenerate cases, the solution is determined up to one indeterminate parameter.*

It can be seen after some manipulations that $\|\boldsymbol{F}^\top\boldsymbol{k}\|$, $\|\boldsymbol{F}\boldsymbol{k}\|$, and $(\boldsymbol{k}, \boldsymbol{F}\boldsymbol{k})$ are linearly related to $\|\bar{\boldsymbol{E}}^\top\boldsymbol{k}\|$, $\|\bar{\boldsymbol{E}}\boldsymbol{k}\|$, and $(\boldsymbol{k}, \bar{\boldsymbol{E}}\boldsymbol{k})$ and vice versa. It follows that $(\|\boldsymbol{F}^\top\boldsymbol{k}\|, \|\boldsymbol{F}\boldsymbol{k}\|, (\boldsymbol{k}, \boldsymbol{F}\boldsymbol{k})) \neq (0,0,0)$ if and only if $(\|\bar{\boldsymbol{E}}^\top\boldsymbol{k}\|, \|\bar{\boldsymbol{E}}\boldsymbol{k}\|, (\boldsymbol{k}, \bar{\boldsymbol{E}}\boldsymbol{k})) \neq (0,0,0)$. On the other hand, eq. (20) implies

$$K_x = 2g'(Z)(cY+1) + 2A, \quad K_y = 2g'(Z)(cY+1) + 2A. \tag{38}$$

Hence, we have $Z = P$ or equivalently $g'(Z) = (Z - P)/2 = 0$ if and only if $A = B = 0$. From eq. (22), this means $g(P) = 0$. But $g(P) = g'(P) = 0$ implies that $g(Z)$ has the form $g(Z) = (Z - P)^2/4$, which in turn implies that eq. (20) has the form

$$K = \frac{1}{2}(XY + X + Y - P)^2. \tag{39}$$

This case corresponds to Case 4 in the above list. Thus, our algorithm fails if and only if the cameras are in a degenerate configuration as described in Proposition 1.

6. SINGLE FOCAL LENGTH SOLUTION

If the cameras are in a degenerate configuration, we can determine f and f' by assuming that they are identical. Letting $f = f'$ in our algorithm, we obtain the following procedure:

1. Compute the following quantities:

$$\begin{aligned}
a_1 &= \frac{1}{2}(\boldsymbol{k}, \boldsymbol{F}\boldsymbol{k})^4, \\
a_2 &= (\boldsymbol{k}, \boldsymbol{F}\boldsymbol{k})^2(\|\boldsymbol{F}^\top\boldsymbol{k}\|^2 + \|\boldsymbol{F}\boldsymbol{k}\|^2), \\
a_3 &= \frac{1}{2}(\|\boldsymbol{F}^\top\boldsymbol{k}\|^2 - \|\boldsymbol{F}\boldsymbol{k}\|^2)^2 \\
&\quad + (\boldsymbol{k}, \boldsymbol{F}\boldsymbol{k})(4(\boldsymbol{k}, \boldsymbol{F}\boldsymbol{F}^\top\boldsymbol{F}\boldsymbol{k}) - (\boldsymbol{k}, \boldsymbol{F}\boldsymbol{k})\|\boldsymbol{F}\|^2), \\
a_4 &= 2(\|\boldsymbol{F}\boldsymbol{F}^\top\boldsymbol{k}\|^2 + \|\boldsymbol{F}^\top\boldsymbol{F}\boldsymbol{k}\|^2) - (\|\boldsymbol{F}^\top\boldsymbol{k}\|^2 \\
&\quad + \|\boldsymbol{F}\boldsymbol{k}\|^2)\|\boldsymbol{F}\|^2, \\
a_5 &= \|\boldsymbol{F}\boldsymbol{F}^\top\|^2 - \frac{1}{2}\|\boldsymbol{F}\|^4.
\end{aligned} \tag{40}$$

2. Define the following polynomial:

$$K(x) = a_1 x^4 + a_2 x^3 + a_3 x^2 + a_4 x + a_5. \tag{41}$$

3. If $a_1 \approx a_2 \approx a_3 \approx 0$, stop. Otherwise, compute the common root of $K(x)$ and $K'(x)$.
4. Return the following focal length f:

$$f = \frac{1}{\sqrt{1+x}}. \tag{42}$$

The common root of $K(x)$ and $K'(x)$ is computed as follows. If $a_1 \neq 0$, the three equations $K(x) = 0$, $K'(x) = 0$, and $xK'(x) = 0$ are written in the form

$$\begin{pmatrix} a_1 & a_2 & a_3x^2 + a_4x + a_5 \\ & 4a_1 & 3a_2x^2 + 2a_3x + a_4 \\ 4a_1 & 3a_2 & 2a_3x^2 + a_4x \end{pmatrix} \begin{pmatrix} x^4 \\ x^3 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{43}$$

Hence, we obtain the following quadratic equation:

$$\begin{aligned}
&\begin{vmatrix} a_1 & a_2 & a_3x^2 + a_4x + a_5 \\ & 4a_1 & 3a_2x^2 + 2a_3x + a_4 \\ 4a_1 & 3a_2 & 2a_3x^2 + a_4x \end{vmatrix} \\
&= a_1(3a_2^2 - 8a_1a_3)x^2 + 2a_1(a_2a_3 - 6a_1a_4)x \\
&\quad + a_1(a_2a_4 - 16a_1a_5) = 0.
\end{aligned} \tag{44}$$

If we obtain two solutions (the above equation may degenerate into a linear equation), we choose the one that satisfies $K(x) = 0$. If $a_1 = 0$ but $a_2 \neq 0$, the three equations $K(x) = 0$, $K'(x) = 0$, and $xK'(x) = 0$ are written in the form

$$\begin{pmatrix} a_2 & a_3 & a_4x + a_5 \\ & 3a_2 & 2a_3x + a_4 \\ 3a_2 & 2a_3 & a_4x \end{pmatrix} \begin{pmatrix} x^3 \\ x^2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{45}$$

Hence, we obtain the following linear equation to determine a single solution:

$$\begin{aligned}
&\begin{vmatrix} a_2 & a_3 & a_4x + a_5 \\ & 3a_2 & 2a_3x + a_4 \\ 3a_2 & 2a_3 & a_4x \end{vmatrix} \\
&= 2a_2(a_3^2 - 3a_2a_4)x + a_2(a_3a_4 - 9a_2a_5) = 0.
\end{aligned} \tag{46}$$

If $a_1 = a_2 = 0$ but $a_3 \neq 0$, we have a linear equation $K'(x) = 0$ to determine a single solution.

The above algorithm works unless $a_1 = a_2 = a_3 = 0$ (we will study this case shortly), because the input

$\boldsymbol{F}$ is assumed to have the single-focal-length solution $f = f'$ either from a priori knowledge or as a result of the degeneracy check of the general algorithm. If we want the above algorithm to work for an arbitrary fundamental matrix $\boldsymbol{F}$ for which $K(x) = K'(x) = 0$ may not have any solution, a reasonable strategy may be to solve the cubic (or lesser degree) equation $K'(x) = 0$ and choose the solution that minimizes $|K(x)|$.

7. FOCAL LENGTH DEGENERACY

We switch to the $x'y'$ coordinate system given by eq. (26) so that the solution is at the origin $(0, 0)$. Letting $f = f'$ in eq. (27), we obtain the polynomial $K(x')$ in form

$$K(x') = \bar{a}_1 x'^4 + \bar{a}_2 x'^3 + \bar{a}_3 x'^2, \qquad (47)$$

where $\bar{a}_1$, $\bar{a}_2$, and $\bar{a}_3$ are the values of a_1, a_2, and a_3, respectively, obtained by replacing $\boldsymbol{F}$ in eqs. (40) by $\bar{\boldsymbol{E}}$ defined by eq. (28). Evidently, $K(0) = K'(0) = 0$. If eqs. (31) are substituted, $K''(0)$ has the following form:

$$K''(0) = \|\boldsymbol{t}\|^4 \Big(4\sin^2\phi \sin^2\theta \sin^2\theta' + (\sin^2\theta - \sin^2\theta')^2\Big). \qquad (48)$$

This vanishes when $\sin^2\phi \sin^2\theta \sin^2\theta' = 0$ and $\sin^2\theta = \sin^2\theta'$.

The former condition means $\phi = 0$, π (the two optical axes and the translation direction are coplanar), $\theta = 0$, π (the first optical axis and the translation direction are collinear), or $\theta' = 0$, π (the second optical axis and the translation direction are collinear).

The latter condition means $\theta' = \theta$, $\theta + \pi$ (the two optical axes are parallel) or $\theta' = -\theta$, $\pi - \theta$ (the two optical axes intersect and form an equilateral triangle with the baseline as its base).

It is easy to confirm that $K(x')$ identically vanish if $\sin^2\phi \sin^2\theta \sin^2\theta' = 0$ and $\sin^2\theta = \sin^2\theta'$. Thus, we obtain the following proposition:

Proposition 3 *If the focal length is fixed, it is indeterminate if and only if (i) the two optical axes are parallel or (ii) the two optical axes intersect and form an equilateral triangle with the baseline as its base.*

Brooks et al. [2] pointed out that the case (ii) was a singular configuration for a stereo head with a horizontally baseline, horizontal optical axes, and vertical axes of camera rotation. Our result has shown that the cases (i) and (ii) are the only possibilities of degeneracy among all unconstrained camera configurations.

8. ACCURACY AND ROBUSTNESS

Our algorithm is, like all others [1, 5, 16, 17, 15], a mapping from the fundamental matrix $\boldsymbol{F}$ to the focal lengths f and f' and the motion parameter $\{\boldsymbol{t}, \boldsymbol{R}\}$. Hence, it does not make sense to compare our algorithm with others with respect to accuracy and robustness: all algorithms produce exactly the same result. The computation time, however, somewhat differs from algorithm to algorithm; because eigenvalue computation and singular value decomposition are not involved, our algorithm seems slightly faster than others, although the speed is very much affected by the machine and implementation.

On the other hand, the accuracy of f, f', and $\{\boldsymbol{t}, \boldsymbol{R}\}$ does not depend on the decomposition algorithm; it depends solely on the accuracy of computing the fundamental matrix $\boldsymbol{F}$. We have already developed a numerical scheme for optimally computing the fundamental matrix from point correspondences in the presence of noise and evaluating the reliability of the computed solution [11]. The algorithm is strictly optimal and is guaranteed to satisfy the theoretical accuracy bound; there is no room for further improvement. Our program is implemented in the C++ language and is publicly available via the Web[1].

9. CONCLUDING REMARKS

We have described an algorithm for decomposing a given fundamental matrix into two focal lengths and motion parameters in a closed-form expression in terms of the elements of the fundamental matrix alone, assuming that other camera parameters are known by pre-calibration. It is based on the decomposability condition of the essential matrix expressed in terms of its scalar invariants. We have given a complete analysis for degenerate camera configurations and obtained the same result as Newsam et al. [15]. Finally, we have given an algorithm for computing a single focal length in the degenerate case. We have shown that the solution is indeterminate only for the singular configurations that Brooks et al. [2] found for a horizontally-constrained stereo head.

In Appendix A, we recapitulate Bougnoux's formula [1] in the framework of this paper in a slightly different from.

Acknowledgments. The authors thank Bill Triggs of INRIA Rône-Alpes, France, Du Huynh of Murdoch University, Australia, and Toshio Ueshiba of Electrotechnical Laboratory, Japan, for helpful discussions. This work was in part supported by the Ministry of Education, Science, Sports and Culture, Japan under a Grant in Aid for Scientific Research C(2) (No. 11680377).

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APPENDIX A: BOUGNOUX'S FORMULA

Let $\boldsymbol{e}$ and $\boldsymbol{e}'$ be the unit eigenvectors of $\boldsymbol{F}^\top$ and $\boldsymbol{F}$, respectively, for eigenvalue zero; they represent the *epipoles* of the first and second images [3, 4, 13, 18]. We can eliminate the translation $\boldsymbol{t}$ by using $\boldsymbol{e}$ and $\boldsymbol{e}'$ and express $\boldsymbol{F}$ in the following two forms:

$$\boldsymbol{F} \simeq \boldsymbol{e} \times \mathrm{diag}(1,1,\frac{1}{f})\boldsymbol{R}\mathrm{diag}(1,1,f'),$$
$$\boldsymbol{F} \simeq \mathrm{diag}(1,1,f)\boldsymbol{R}\mathrm{diag}(1,1,\frac{1}{f'}) \times \boldsymbol{e}'. \quad (49)$$

Here, $\simeq$ means that one side is a multiple of the other by a nonzero constant. For a matrix $\boldsymbol{T}$ and a vector $\boldsymbol{u}$, we define $\boldsymbol{T} \times \boldsymbol{u}$ to be $\boldsymbol{T}(\boldsymbol{u} \times \boldsymbol{T})^\top$. Eliminating $\boldsymbol{R}$ from the above equations, we obtain the following *Kruppa equations* [3, 6, 13, 14]:

$$\boldsymbol{F}\mathrm{diag}(1,1,\frac{1}{f'^2})\boldsymbol{F}^\top \simeq \boldsymbol{e} \times \mathrm{diag}(1,1,\frac{1}{f^2}) \times \boldsymbol{e},$$
$$\boldsymbol{F}^\top\mathrm{diag}(1,1,\frac{1}{f^2})\boldsymbol{F} \simeq \boldsymbol{e}' \times \mathrm{diag}(1,1,\frac{1}{f'^2}) \times \boldsymbol{e}'. \quad (50)$$

In terms of x and y defined by eqs. (16), these equations are rewritten as

$$\boldsymbol{F}(\boldsymbol{I} + y\boldsymbol{k}\boldsymbol{k}^\top)\boldsymbol{F}^\top \simeq \boldsymbol{e} \times (\boldsymbol{I} + x\boldsymbol{k}\boldsymbol{k}^\top) \times \boldsymbol{e},$$
$$\boldsymbol{F}^\top(\boldsymbol{I} + x\boldsymbol{k}\boldsymbol{k}^\top)\boldsymbol{F} \simeq \boldsymbol{e}' \times (\boldsymbol{I} + y\boldsymbol{k}\boldsymbol{k}^\top) \times \boldsymbol{e}', \quad (51)$$

from which we obtain

$$\boldsymbol{F}\boldsymbol{F}^\top + y(\boldsymbol{F}\boldsymbol{k})(\boldsymbol{F}\boldsymbol{k})^\top \simeq \boldsymbol{P}_{\mathbf{e}} + x(\boldsymbol{e}\times\boldsymbol{k})(\boldsymbol{e}\times\boldsymbol{k})^\top, \quad (52)$$

$$\boldsymbol{F}^\top\boldsymbol{F} + x(\boldsymbol{F}^\top\boldsymbol{k})(\boldsymbol{F}^\top\boldsymbol{k})^\top \simeq \boldsymbol{P}_{\mathbf{e}'} + y(\boldsymbol{e}'\times\boldsymbol{k})(\boldsymbol{e}'\times\boldsymbol{k})^\top, \quad (53)$$

where

$$\boldsymbol{P}_{\mathbf{e}} = \boldsymbol{I} - \boldsymbol{e}\boldsymbol{e}^\top, \qquad \boldsymbol{P}_{\mathbf{e}'} = \boldsymbol{I} - \boldsymbol{e}'\boldsymbol{e}'^\top. \quad (54)$$

Multiplying $\boldsymbol{k}$ from the right on both sides of eqs. (52) and (53), we obtain

$$\boldsymbol{F}\boldsymbol{F}^\top\boldsymbol{k} + y(\boldsymbol{k},\boldsymbol{F}\boldsymbol{k})\boldsymbol{F}\boldsymbol{k} = c\boldsymbol{P}_{\mathbf{e}}\boldsymbol{k},$$
$$\boldsymbol{F}^\top\boldsymbol{F}\boldsymbol{k} + x(\boldsymbol{k},\boldsymbol{F}\boldsymbol{k})\boldsymbol{F}^\top\boldsymbol{k} = c'\boldsymbol{P}_{\mathbf{e}'}\boldsymbol{k}, \quad (55)$$

where c and c' are unknown constants. Taking the inner product of $\boldsymbol{k}$ and both sides of the second of eqs. (55), we obtain

$$\|\boldsymbol{F}\boldsymbol{k}\|^2 + (\boldsymbol{k},\boldsymbol{F}\boldsymbol{k})^2 x = c'\|\boldsymbol{e}'\times\boldsymbol{k}\|^2. \quad (56)$$

Taking the inner product of $\boldsymbol{F}^\top\boldsymbol{k}$ and both sides of the second of eqs. (55), we obtain

$$(\boldsymbol{k},\boldsymbol{F}\boldsymbol{F}^\top\boldsymbol{F}\boldsymbol{k}) + (\boldsymbol{k},\boldsymbol{F}\boldsymbol{k})\|\boldsymbol{F}^\top\boldsymbol{k}\|^2 x = c'(\boldsymbol{k},\boldsymbol{F}\boldsymbol{k}). \quad (57)$$

Eqs. (56) and (57) can be solved for x in the form

$$x = \frac{\|\boldsymbol{F}\boldsymbol{k}\|^2 - (\boldsymbol{k},\boldsymbol{F}\boldsymbol{F}^\top\boldsymbol{F}\boldsymbol{k})\|\boldsymbol{e}'\times\boldsymbol{k}\|^2/(\boldsymbol{k},\boldsymbol{F}\boldsymbol{k})}{\|\boldsymbol{e}'\times\boldsymbol{k}\|^2\|\boldsymbol{F}^\top\boldsymbol{k}\|^2 - (\boldsymbol{k},\boldsymbol{F}\boldsymbol{k})^2}. \quad (58)$$

Similarly, we obtained from the first of eqs. (55)

$$y = \frac{\|\boldsymbol{F}^\top\boldsymbol{k}\|^2 - (\boldsymbol{k},\boldsymbol{F}\boldsymbol{F}^\top\boldsymbol{F}\boldsymbol{k})\|\boldsymbol{e}\times\boldsymbol{k}\|^2/(\boldsymbol{k},\boldsymbol{F}\boldsymbol{k})}{\|\boldsymbol{e}\times\boldsymbol{k}\|^2\|\boldsymbol{F}\boldsymbol{k}\|^2 - (\boldsymbol{k},\boldsymbol{F}\boldsymbol{k})^2}. \quad (59)$$

Thus, f and f' are given by

$$f = \frac{1}{\sqrt{1+x}}, \qquad f' = \frac{1}{\sqrt{1+y}}. \quad (60)$$

This result is essentially the same as the formula for f given by Bougnoux [1], although the appearance is slightly different.

APPENDIX B: SINGULARITY OF $\boldsymbol{E}$

Since the scale of the essential matrix $\boldsymbol{E}$ is indeterminate, we can assume that $\|\boldsymbol{E}\| = 1$ without loosing generality. This means that $\boldsymbol{E}$ is a point on an 8-dimensional unit sphere S^8 centered at the origin in the 9-dimensional parameter space. Eq. (13) states that the true value $\bar{\boldsymbol{E}}$ is an intersection of the sphere S^8 with the manifold defined by $\|\boldsymbol{E}\boldsymbol{E}^\top\|^2 = 1$. Let $\bar{\boldsymbol{E}} + \Delta\boldsymbol{E}$ be a neighboring point to $\bar{\boldsymbol{E}}$ on the manifold. To a first approximation, we obtain

$$1 = \|(\bar{\boldsymbol{E}}+\Delta\boldsymbol{E})(\bar{\boldsymbol{E}}+\Delta\boldsymbol{E})^\top\|^2$$
$$= 1 + 2(\bar{\boldsymbol{E}}\bar{\boldsymbol{E}}^\top\bar{\boldsymbol{E}};\Delta\boldsymbol{E}), \quad (61)$$

where we define the inner product of matrices $\boldsymbol{A} = (A_{ij})$ and $\boldsymbol{B} = (B_{ij})$ by $(\boldsymbol{A};\boldsymbol{B}) = \sum_{i,j=1}^3 A_{ij}B_{ij}$. The above equation implies that the "surface normal" to the manifold $\|\boldsymbol{E}\boldsymbol{E}^\top\|^2 = 1$ at $\bar{\boldsymbol{E}}$ is $\bar{\boldsymbol{E}}\bar{\boldsymbol{E}}^\top\bar{\boldsymbol{E}}$. Substituting eq. (29), we obtain

$$\bar{\boldsymbol{E}}\bar{\boldsymbol{E}}^\top\bar{\boldsymbol{E}} = \|\boldsymbol{t}\|^2\bar{\boldsymbol{E}}. \quad (62)$$

Since the manifold normal to the sphere S^8 at $\bar{\boldsymbol{E}}$ is $\bar{\boldsymbol{E}}$ itself, the above equation implies that the sphere S^8 is *tangent* to the manifold $\|\boldsymbol{E}\boldsymbol{E}^\top\|^2 = 1$ at $\bar{\boldsymbol{E}}$, sharing a common tangent space there. If $\bar{\boldsymbol{E}}$ is constrained to be in a two-parameter subset of the sphere S^8 parameterized by f and f', the tangency relation also holds for arbitrary perturbations of f and f'. Hence eq. (15) holds.

Since $\|\boldsymbol{E}\boldsymbol{E}^\top\|^2 = 1$ and S^8 are eight-dimensional as manifolds, their intersection in the nine-dimensional space should be a seven-dimensional manifolds in general. Because of the above *non-transversality*, however, the intersection is a six-dimensional manifold. The solution $\bar{\boldsymbol{E}}$ is in the intersection of this six-dimensional manifold with the eight-dimensional manifold defined by $\det\boldsymbol{E} = 0$. It follows that the intersection is a five-dimensional manifold. Thus, the essential matrix has five degrees of freedom corresponding to the camera rotation and the normalized translation.