Int. J. Engng Sci. Vol. 20, No. 7, pp. 879–884, 1982
Printed in Great Britain.

0020-7225/82/070879–06$03.00/0

DILATANT PLASTIC DEFORMATION OF GRANULAR MATERIALS

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Abstract—A geometrical interpretation is given to the modified associated flow rule derived in the previous paper[1]. According to it, the dilatancy must be regarded as an internal constraint of deformation. The modified associated flow rule then gives equations of plastic deformation which exhibits the specified dilatancy. Hardening and elastic strains can also be incorporated. It is shown that the deformation is non-coaxial in general.

1. INTRODUCTION

IN THE previous paper[1], we re-examined Drucker's plastic work postulate and showed that a mechanically consistent plastic work postulate can be obtained for granular materials with frictional stresses if internal constraints of deformation are taken into account. We derived from it a modified version of the associated flow rule and applied it to nondilatant deformation of granular materials. Then, characteristic surfaces and singular wave propagation were analyzed. An extensive review on the background of the subject was also given there.

The deformation of granular materials is classified into several stages. First, under the external forces the material begins an incipient dilatant deformation. Then, it passes a critical state and goes into a flow regime without dilatancy. It is this last stage that the previous theory[1] describes. In this paper, we consider the incipient dilatant plastic deformations of granular materials, employing the same principle.

We first give a geometrical interpretation of the modified associated flow rule obtained in the previous paper[1]. As was discussed there, the dilatancy must be regarded as an internal constraint of deformation. In other words, the flow rule applies to deformations other than the dilatancy, and the dilatancy is not expected to be derived from the flow rule[1]. Applying the modified associated flow rule to the constraint, we derive equations of plastic deformations that exhibit the specified dilatancy. Hardening and elastic strains can also be incorporated. It is shown that the deformation is non-coaxial in general.

Today, as was discussed in [1], various types of yield functions, plastic potentials and hardening rules have been elaborated for the fitting of experimental data. A number of *ad hoc* assumptions are made without a theoretically sound basis, introducing a large number of indeterminate material constants. Then, the analyses based on them are shown to agree with experimental observations by a clever choice of the values for the material constants. However, introduction of a large number of *ad hoc* assumptions, though it helps to gain experimental support, often hinders our understanding of mechanical laws essential to the phenomenon. In contrast, our theory is very simple in the sense that all the material constants involved are α and k specifying the yield condition plus the dilatancy factor β alone, and is based on a reasonable mechanical foundation[1]. Yet, it illuminates mechanical features characteristic to granular materials. The theory is a natural extension of the previous one[1], and all the equations are expressed in terms of three-dimensional Cartesian tensor equations.

2. GEOMETRICAL INTERPRETATION OF THE MODIFIED ASSOCIATED FLOW RULE

We now give a geometrical interpretation of the associated flow rule derived from a new plastic work postulate in [1]. Suppose there is a constraint of deformation. For example, if the material is incompressible, or *nondilatant* in the terminology of soil mechanics, we have

$$e_{kk} = 0, \tag{2.1}$$

where e_{ij} is the strain tensor. We henceforth adopt the Cartesian tensor notation and the summation convention. Now, consider a constraint of deformation expressed as a single equation

$$F(e_{ji}) = 0, \tag{2.2}$$

which gives a hypersurface in the strain space.

Next, consider the associated constraining stress. It is a part of the stress which does not do any work for admissible deformations but does work only for virtual deformations that violate the constraint. Let an infinitesimal deformation $e_{ij} \to e_{ji} + \mathrm{d}e_{ji}$ be admissible. Then,

$$F(e_{ji}) = F(e_{ji} + \mathrm{d}e_{ji}) = 0, \tag{2.3}$$

and hence

$$\frac{\partial F}{\partial e_{ji}} \mathrm{d}e_{ji} = 0, \tag{2.4}$$

which means that $\partial F/\partial e_{ji}$ is a vector *normal* to the hypersurface in the strain space. If σ^c_{ji} is the constraining stress, it does not do work for the strain increment and hence

$$\sigma^c_{ji}\, \mathrm{d}e_{ji} = 0. \tag{2.5}$$

Now, identify the stress space with the strain space, using the same coordinate axis for the components of the same indices pair. Comparing eqn (2.4) with eqn (2.5), we can see that σ^c_{ji} is *parallel* to $\partial F/\partial e_{ji}$. Let σ_{ji} be the total stress and let n_{ji} is $\partial F/\partial e_{ji}$ multiplied by a constant such that $n_{ji}n_{ji} = 1$. The magnitude ξ of the component of σ_{ji} in the direction of the *unit* vector n_{ji} is

$$\xi = n_{ji}\sigma_{ji}. \tag{2.6}$$

The region in the stress space in which the constraining stress is constant is a hyperplane whose equation is given by eqn (2.6). The constraining stress is the normal drawn to the hyperplane from the origin (Fig. 1) and is given by

$$\sigma^c_{ji} = \xi n_{ji}. \tag{2.7}$$

This constraining stress depends not only on the total stress σ_{ji} but also on the strain e_{ji}, because the hypersurface of the constraint is *curved* in general and hence the direction of the surface normal depends on the present strain. If we consider the constraint of incompressibility or nondilatancy, then

$$F(e_{ji}) = e_{kk}, \tag{2.8}$$

and the hypersurface is *flat*. Since $\partial F/\partial e_{ji} = \delta_{ji}$ (Kronecker delta), the unit normal is

$$n_{ji} = \delta_{ji}/\sqrt{3} \tag{2.9}$$

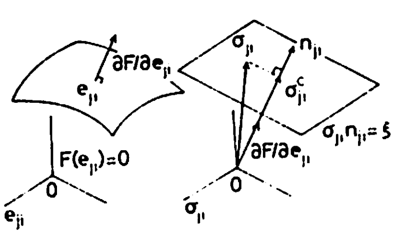

Fig. 1.

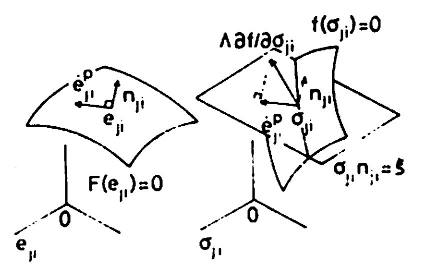

Fig. 2.

Fig. 1. The constraint of deformation and the constraining stress.

Fig. 2. Geometrical interpretation of the modified associated flow rule.

and the magnitude of the constraining stress is

$$\xi = \sigma_{kk}/\sqrt{3}\ (= -\sqrt{(3)}p), \tag{2.10}$$

where $p = -\sigma_{kk}/3$ is the hydrostatic pressure. Thus, the constraining stress for the constraint of incompressibility is

$$\sigma^c_{ji} = -p\delta_{ji}, \tag{2.11}$$

as is expected.

On the other hand, consider a yield condition in the form of

$$f(\sigma_{ji}) = 0, \tag{2.12}$$

which gives a hypersurface, i.e. the yield surface, in the stress space. Now, take a new Cartesian coordinate system in the stress space such that one axis is in the direction of the unit normal n_{ji} of the surface of constraint. Let $\sigma_{\beta\alpha}$ be the components of the stress with respect to the remaining coordinate axes and express the yield condition in terms of ξ and $\sigma_{\beta\alpha}$

$$f(\sigma_{\beta\alpha}, \xi) = 0. \tag{2.13}$$

As was discussed in [1], the new plastic work postulate leads to the modified associated flow rule

$$\dot{e}^p_{ji} = \Lambda \left.\frac{\partial f(\sigma_{\beta\alpha}(\sigma_{ji}), \xi)}{\partial \sigma_{ji}}\right|_{\xi=\text{const.}}, \tag{2.14}$$

where $\dot{e}^p_{ji}$ is the plastic strain-rate and Λ is a scalar quantity. In [1], we considered the constraint of incompressibility. There, the hydrostatic pressure p was used instead of ξ by virtue of eqn (2.10), and $\sigma_{\beta\alpha}$ was identified with the stress deviator

$$\bar{\sigma}_{ji} = \sigma_{ji} - \tfrac{1}{3}\sigma_{kk}\delta_{ji}, \tag{2.15}$$

since $\bar{\sigma}_{ji}$ is *perpendicular* to $p\delta_{ji}$, i.e. $p\delta_{ji}\bar{\sigma}_{ji} = 0$.

Consider a geometrical interpretation of eqn (2.14). Let us consider a hyperplane in the stress space passing the present stress σ_{ji} and having a surface normal parallel to n_{ji}. (The stress space and the strain space are identified as before.) Then, eqn (2.14) implies that the plastic strain-rate $\dot{e}^p_{ji}$ is *parallel to the hyperplane, where the constraining stress is constant, and perpendicular to the intersection of the hyperplane and the yield surface.* Thus, in order to obtain the direction of $\dot{e}^p_{ji}$, we may first construct the normal $\partial f/\partial\sigma_{ji}$ of the yield surface and then *project* it onto the hyperplane (Fig. 2). The strain-rate so obtained is parallel to the *tangent plane* to the surface of the constraint at the present strain e_{ji} in the strain space. This is an obvious requirement for any possible strain-rate not to violate the constraint.

3. PLASTIC DEFORMATION OF DILATANT MATERIALS

As was discussed in [1], the plastic flow originates in microscopic *slips* and hence is incompressible in nature. The dilatancy, i.e. the non-elastic change in specific volume of granular materials occurs due to the fact that the material consists of particles in contact with each other. Hence, the dilatancy is geometrical in nature and, therefore, must be regarded as an internal constraint of deformation. In other words, the plastic flow rule applies to deformations other than the dilatancy, and the dilatancy is not expected to be derived from the flow rule[1].

The dilatancy depends on the internal packing configuration of the constituent particles, which in turn is determined not only by the type of the material but also by the past history of deformation. Here, we restrict our theory to a given initial state of the material and only consider small deformations without unloading. In this range of deformation, we can approximate the dilatancy relation by

$$v = \beta\gamma_s, \tag{3.1}$$

where v is the volumetric strain and γ_s is the shear strain. We call the constant β the dilatancy factor, which takes on a value determined by the type of the material and its past history of deformation. Of course, we can assume a more complicated relation and derive a theory similar to the subsequent one. The validity of the law (3.1) must be judged from the objects and the purposes of particular applications of the theory.

In the Cartesian tensor notation, the volumetric strain v and the shear strain γ_s for general deformations are given, respectively, by

$$v = e_{kk}, \qquad \gamma_s = \sqrt{(\bar{e}_{ji}\bar{e}_{ji}/2)}, \tag{3.2}$$

where

$$\bar{e}_{ji} = e_{ji} - \tfrac{1}{3}e_{kk}\delta_{ji} \tag{3.3}$$

is the strain deviator. Hence, the constraint of deformation is

$$e_{kk} = \beta\sqrt{(\bar{e}_{ji}\bar{e}_{ji}/2)}, \tag{3.4}$$

i.e.

$$F(e_{ji}) = e_{kk} - \beta\sqrt{(\bar{e}_{ji}\bar{e}_{ji}/2)}. \tag{3.5}$$

The unit normal to the hypersurface of the constraint is

$$n_{ji} = \frac{1}{\sqrt{(3+\beta^2/2)}}\left(\delta_{ji} - \frac{\beta^2}{2}\frac{\bar{e}_{ji}}{e_{kk}}\right). \tag{3.6}$$

The magnitude of the constraining stress is given by

$$\xi = -\sqrt{3}\left(p + \frac{\beta^2}{6}\frac{\bar{\sigma}_{ji}\bar{e}_{ji}}{e_{kk}}\right). \tag{3.7}$$

As the yield condition we adopt the extended von Mises equation

$$\sqrt{(\bar{\sigma}_{ji}\bar{\sigma}_{ji}/2)} = \alpha p + k. \tag{3.8}$$

following Drucker–Prager[2] and the previous paper[1]. (The validity of this equation is also discussed in [1].) The normal to the yield surface is

$$\frac{\partial f}{\partial\sigma_{ji}} = \frac{\bar{\sigma}_{ji}}{2(\alpha p + k)} + \frac{\alpha}{3}\delta_{ji}. \tag{3.9}$$

The magnitude of the component of this vector projected in the direction of n_{ji} is

$$n_{ji}\frac{\partial f}{\partial\sigma_{ji}} = \frac{1}{\sqrt{(3+\beta^2/2)}}\left(\alpha - \frac{\beta^2}{4}\frac{\bar{\sigma}_{ji}\bar{e}_{ji}}{(\alpha p + k)e_{kk}}\right). \tag{3.10}$$

The component of the vector (3.9) perpendicular to n_{ji} is $\partial f/\partial\sigma_{ji} - (n_{lk}\partial f/\partial\sigma_{lk})n_{ji}$, and hence

$$\dot{e}^p_{ji} \propto \partial f/\partial\sigma_{ji} - (n_{lk}\partial f/\sigma_{lk})n_{ji}. \tag{3.11}$$

In terms of the deviator part and the scalar part, this becomes

$$\dot{e}^p_{ji} = \Lambda\left[\frac{1+6/\beta^2}{2}\frac{\bar{\sigma}_{ji}}{\alpha p + k} + \left(\alpha - \frac{\beta^2}{4}\frac{\bar{\sigma}_{lk}\bar{e}_{lk}}{(\alpha p + k)e_{mm}}\right)\frac{\bar{e}_{ji}}{e_{nn}}\right], \tag{3.12}$$

$$\dot{e}^p_{kk} = \Lambda\left[\alpha + \frac{3}{2}\frac{\bar{\sigma}_{ji}\bar{e}_{ji}}{(\alpha p + k)e_{kk}}\right], \tag{3.13}$$

where Λ is a scalar quantity. It is easily checked that these equations are compatible with the strain constraint (3.4).

Following the usual routine process, we can also incorporate hardening and elastic strains into the theory. Without going into detail, we only show the final results. Let the total strain-rate be resolved into the elastic part $\dot{e}^e_{ji}$ and the plastic part $\dot{e}^p_{ji}$ and assume linearity

$$\frac{D\sigma_{ji}}{Dt} = 2\mu\dot{e}^e_{ji} + \lambda\dot{e}^e_{kk}\delta_{ji}, \tag{3.14}$$

where μ and λ are constants and D/Dt is the Jaumann–Noll derivative[1]. Adopting the isotropic hardening law

$$\sqrt{(\bar{\sigma}_{ji}\bar{\sigma}_{ji}/2)} = \alpha(e)p + k(e), \tag{3.15}$$

where e is the void ratio, we finally have

$$\frac{D\bar{\sigma}_{ji}}{Dt} = 2\mu\left[\dot{\bar{e}}_{ji} - \Lambda\left\{\frac{1+6/\beta^2}{2}\frac{\bar{\sigma}_{ji}}{\alpha p + k} + \left(\alpha - \frac{\beta^2}{4}\frac{\bar{\sigma}_{lk}\bar{e}_{lk}}{(\alpha p + k)e_{mm}}\right)\frac{\bar{e}_{ji}}{e_{nn}}\right\}\right], \tag{3.16}$$

$$\frac{dp}{dt} = -\kappa\left[\dot{e}_{kk} - \Lambda\left\{\alpha + \frac{3}{2}\frac{\bar{\sigma}_{ji}\bar{e}_{ji}}{(\alpha p + k)e_{kk}}\right\}\right], \tag{3.17}$$

$$\frac{de}{dt} = (1+e)\Lambda\left[\alpha + \frac{3}{2}\frac{\bar{\sigma}_{ji}\bar{e}_{ji}}{(\alpha p + k)e_{kk}}\right]. \tag{3.18}$$

where we have put $\kappa = (2\mu + 3\lambda)/3$,

$$\Lambda = \frac{\mu\bar{\sigma}_{ji}\dot{\bar{e}}_{ji}/(\alpha p + k) + \alpha\kappa\dot{e}_{kk} - (\alpha' p + k')\,de/dt}{\left(1+\frac{6}{\beta^2}\right)\mu + \alpha^2\kappa + \frac{\bar{\sigma}_{ji}\bar{e}_{ji}}{(\alpha p + k)e_{kk}}\left\{\alpha\left(\mu + \frac{3}{2}\lambda\right) - \frac{\mu\beta^2}{4}\frac{\bar{\sigma}_{lk}\bar{e}_{lk}}{(\alpha p + k)e_{mm}}\right\}}, \tag{3.19}$$

and $\alpha' = d\alpha/de$, $k' = dk/de$.

There is plenty of room for possible plausible assumptions about eqns (3.14) and (3.15). However, we do not go into it further, because our purpose is to illustrate the geometrical nature of the dilatancy and not the experimental data fitting, as we have stated earlier.

4. CONDITION OF PLANE DEFORMATION

Let us briefly examine the expression of our theory for usual configurations of experiments. In the following we do not consider hardening and elastic strains. The dilatancy factor β is most easily measured by a triaxial compression test. If the strain is such that $e_{xx} = e_{yy}$, $e_{zz} = -\epsilon$ and other components are zero, then eqn (3.4) is expressed in terms of the volumetric strain $v = 2e_{xx} - \epsilon$ in the form

$$v = \frac{3}{2\sqrt{(3)} - \beta}\epsilon. \tag{4.1}$$

The effect of β on the plastic flow is most easily exhibited in the case of plane deformation. Let the coordinate axes coincide with the principal stress axes and let σ_1, σ_2 and σ_3 be the principal stress components. The deviator components are $\bar{\sigma}_\alpha = \sigma_\alpha + p$ $(\alpha = 1, 2, 3)$, where $p = -(\sigma_1 + \sigma_2 + \sigma_3)/3$. Suppose the principal strain axes rotate counterclockwise relative to the principal stress axes by θ in the x–y plane. Then, according to our theory, plane deformations occur if and only if

$$\eta = \frac{6(1/\beta^2 - 1/12) + \sqrt{(1/\beta^2 - 1/12)}\cos 2\theta}{6(1/\beta^2 - 1/12) - \sqrt{(1/\beta^2 - 1/12)}\cos 2\theta}, \tag{4.2}$$

where we have put $\eta = -\bar{\sigma}_1/\bar{\sigma}_2$, the ratio of the stress deviator components. In order that the plane deformation is coaxial, i.e. $\theta = 0$, the ratio η must take on the value

$$\eta_0 = \frac{6\sqrt{(1/\beta^2 - 1/12)} + 1}{6\sqrt{(1/\beta^2 - 1/12)} - 1}. \tag{4.3}$$

On the other hand, if $1 \leq \eta < \eta_0$, corresponding plane deformations are not coaxial as is shown in Fig. 3. The non-coaxiality is one of the consequences of the fact that the strain-rate depends not only on the stress but also on the strain itself (i.e. the *curved* constraint).

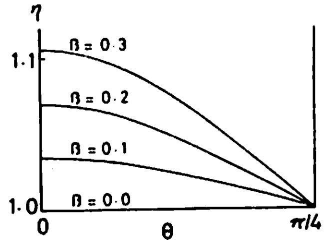

Fig. 3. Relation between the stress deviator ratio and the non-coaxiality of deformation.

If we determine the internal angle of friction ϕ and the cohesion constant c in such a way that the yield condition coincides with the Coulomb condition expressed in terms of ϕ and c when the deformation is plane, we obtain after some calculations

$$\alpha = \frac{2\sqrt{(1 - \eta + \eta^2)}\sin\phi}{1 + \eta - (1 - \eta)\sin\phi}, \qquad k = \frac{2c\sqrt{(1 - \eta + \eta^2)}\cos\phi}{1 + \eta - (1 - \eta)\sin\phi}. \tag{4.4}$$

REFERENCES

[1] K. KANATANI, *Int. J. Engng Sci* 20, 1 (1982).
[2] D. C. DRUCKER and W. PRAGER, *Q. Appl. Math.* 10, 157 (1952).

(*Received* 20 *August* 1981)