COORDINATE ROTATION INVARIANCE OF IMAGE CHARACTERISTICS FOR 3D SHAPE AND MOTION RECOVERY

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ABSTRACT

The values of the parameters describing the shape and motion of an object in a scene are determined from projected images by solving a set of equations relating observed image characteristics to the object parameters. The equations are usually non-linear and are difficult to solve analytically. In these equations, the coordinate system taken on the image plane plays only an auxiliary role, since there exists no coordinate system inherent to the image and any coordinate system can be used equivalently. In this paper, observed image characteristics are shown to be rearranged into invariants with respect to coordinate rotations. Writing the equations of 3D recovery in terms of these invariants makes clear the geometrical meaning of the described relationships, and often analytical solutions emerge themselves. This is illustrated for the analysis of optical flow and the shape-from-texture problem for curved surfaces.

1. INTRODUCTION

One of the objectives of image understanding and computer vision is to reconstruct the 3D shape and motion of an object in a scene from its 2D projection on the image plane. In general terms, the problem is stated as follows.

Suppose an object model is assumed, which is specified by a finite number of parameters $\alpha_1, \ldots, \alpha_n$. For example, if the object is a planar surface, they may be the coordinates of one point on the surface and the two gradient components. If the object is in motion, they may include the translation velocity and the rotation velocity. Let us call these parameters *object parameters*.

Given a projection of the object, we measure some characteristics of the observed image. Let $c_1, \ldots, c_m$ be the data obtained from the measurement. Let us call them the *image characteristics*. They may reflect the gray-levels of the image, the texture of the object surface, the object contour image, the intensity of light reflectance or shading, the optical flow if the object is in motion, etc. Our objective is to compute or estimate the object parameters $\alpha_1, \ldots, \alpha_n$ from the observed image characteristics $c_1, \ldots, c_m$. The problem is often referred to as "shape from ..." depending on the source of the image characteristics, e.g., "shape from texture", "shape from shading", "shape from motion", etc.

If a parameterized object model is assumed and a camera model is known, the image characteristics to be observed can be predicted from the geometry of perspective projection in terms of the object parameters, say

$$c_i = F_i(\alpha_1, \ldots, \alpha_n), \qquad i=1,\ldots,m, \tag{1.1}$$

which we call the *3D recovery equations*. Then, what remains is to solve them and determine the unknown object parameters $\alpha_1, \ldots, \alpha_n$ in terms of the observed image characteristics $c_1, \ldots, c_m$.

Unfortunately, in most cases, the 3D recovery equations are non-linear and are difficult to solve analytically. In the following, we show a way to find analytical solutions, although this may not be always successful. A basic principle is to exploit the fact that the 3D recovery equations have some *geometrical structure* reflecting the geometry of perspective projection. Although there exists no systematic way to solve "arbitrarily given" non-linear equations, we may be able to solve those which have some internal structure.

A way to exploit the underlying geometrical structure is to focus on the *invariant properties* of the object parameters and the image characteristics. For example, note that the coordinate system taken on the image plane plays only an auxiliary role, since there exists no coordinate system inherent to the image and any coordinate system can be used equivalently. If the 3D recovery equations are expressed in terms of *invariants*, in many cases the solutions emerge themselves.

The merit of using invariants is not simply limited to obtaining analytical solutions. As will be shown, the geometrical structure of the problem and the geometrical meaning of involved parameters become very clear if they are expressed in terms of invariants. This will be illustrated in examples of optical flow analysis and the shape-from-texture problem.

2. COORDINATE ROTATION AND REPRESENTATIONS

Suppose a set of image characteristics $c_1, \ldots, c_m$ characterizing a given image is measured in reference to a given Cartesian xy-coordinate system whose origin coincides with the camera optical axis orientation. Since the image itself does not have any inherent coordinate system, the orientation of the x- or y-axis is completely arbitrary. Hence, we may as well use another $x'y'$-coordinate system obtained by rotating the original xy-coordinate system by angle θ counterclockwise.

Let $c_1', \ldots, c_n'$ be new image characteristics obtained by the same measurement but in reference to the new $x'y'$-coordinate system. In many cases, the new values $c_1', \ldots, c_n'$ are related to the original values $c_1, \ldots, c_n$ *linearly* in the form

$$\begin{bmatrix} c_1' \\ \cdot \\ \cdot \\ \cdot \\ c_n' \end{bmatrix} = \begin{bmatrix} & & \\ & T(\theta) & \\ & & \end{bmatrix} \begin{bmatrix} c_1 \\ \cdot \\ \cdot \\ \cdot \\ c_n \end{bmatrix}, \qquad (2.1)$$

or briefly $c' = T(\theta)c$.

It is immediately seen that the coefficient matrix $T(\theta)$ defines a *representation* of the two dimensional rotation group $SO(2)$. In other words, the correspondence from a rotation of the coordinate system by angle θ to matrix $T(\theta)$ defines a *homomorphism* from the 2D rotation group $SO(2)$ into the group of matrices. This is easily shown as follows. Consider another $x''y''$-coordinate system obtained by rotating the $x'y'$-coordinate system by angle θ' counterclockwise. Then, we obtain $c'' = T(\theta')c'$, and hence $c'' = T(\theta')T(\theta)c$. On the other hand, the $x''y''$-coordinate system is obtained by rotating the xy-coordinate system by angle $\theta' + \theta$, and hence we have $c'' = T(\theta'+\theta)c$. As a result, we conclude that $T(\theta')T(\theta) = T(\theta'+\theta)$. Thus, a composition of rotations corresponds to matrix multiplications, and hence $T(\theta)$ defines a homomorphism from rotations into matrices.

Since image characteristics are obtained by particular measurement methods, each of them does not necessarily have a definite meaning. Hence, there exist infinitely many ways of choosing equivalent parameters. For example, instead of using parameters c_1 and c_2, we can equivalently use new parameters $C_1 = c_1 + c_2$ and $C_2 = c_1 - c_2$. The two sets of parameters $\{c_1, c_2\}$, $\{C_1, C_2\}$ describe one and the same property of the image

Suppose we use new image characteristics $C_1, \ldots, C_n$ obtained by taking linear combinations of $c_1, \ldots, c_n$.* Consider the transformation of new parameters $C_1, \ldots, C_n$ due to coordinate rotation by angle θ. Suppose it takes the form

$$\begin{bmatrix} C_1' \\ \vdots \\ C_l' \\ C_{l+1}' \\ \vdots \\ C_n' \end{bmatrix} = \left[\begin{array}{c|c} * & 0 \\ \hline 0 & * \end{array}\right] \begin{bmatrix} C_1 \\ \vdots \\ C_l \\ C_{l+1} \\ \vdots \\ C_n \end{bmatrix} \qquad (2.2)$$

for any θ. Then, the two sets of image characteristics $\{C_1, \ldots, C_l\}$, $\{C_{l+1}, \ldots, C_n\}$ are transformed independently from one another: there exists no mutual coupling. Consequently, we may conclude that *these two sets describe different properties of the image.*

This process of decoupling is called the *reduction of the representation*: the representation $T(\theta)$ is *reduced* to the *direct sum* of two representations. If a representation can be reduced, the representation is said to be *reducible*. In the above case, we may be able to apply the same process of reduction to each of the two sets $\{C_1, \ldots, C_l\}$ and $\{C_{l+1}, \ldots, C_n\}$, then to each of the resulting sets, and so on until no further reduction is possible. Then, we may end up with the form

$$\begin{bmatrix} C_1' \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ C_n' \end{bmatrix} = \begin{bmatrix} * & | & & & & & \\ - & + & - & & & & \\ & | & * & | & & & \\ & & - & + & - & & \\ & & & | & \cdot & & \\ & & & & & \cdot & \\ & & & & & & \cdot \quad | \\ & & & & & & - + - \\ & & & & & & | \quad * \end{bmatrix} \begin{bmatrix} C_1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ C_n \end{bmatrix}. \qquad (2.3)$$

Then, we say that the representation is reduced to the direct sum of *irreducible representations*. If a representation can be reduced to the direct sum of irreducible representations, the representation is said to be *fully reducible* (cf. Hammermesh[1], Kanatani[8]).

If a representation is irreducible, the parameters cannot be separated into independently transforming subsets, no matter what linear combinations are taken. Hence, it is natural to think that such a set of parameters describes a *single* property of the image, whereas a set defining a reducible representation describes two or more different properties simultaneously. In this way, the vague notion of "separating into individual properties" can be given a rigorous mathematical definition as irreducible reduction of a representation with respect to some transformation group. This viewpoint was presented by Hermann Weyl, who asserted that a set of measurement data or *observables* can be regarded as describing a single physical property only if it corresponds to an irreducible representation of a transformation group which does not change the meaning of the phenomenon.[15,16] From this viewpoint, he described quantum mechanics in terms of group representation theory.[16] Let us tentatively call this viewpoint *Weyl's thesis*.

3. INVARIANTS AND WEIGHTS

It is known that for coordinate rotations *all representations are fully reducible* and *all irreducible representations are one-dimensional*. In other words, given image characteristics $c_1, \ldots, c_n$ which define a representation, we can always obtain, by taking appropriate linear combinations, a new set of parameters $C_1, \ldots, C_n$ such that each is

* We assume that the original image characteristics $c_1, \ldots, c_n$ are real numbers. However, we allow the coefficients of these linear combinations to be complex numbers, so that the new image characteristics $C_1, \ldots, C_n$ may be complex numbers.

transformed separately: $C_i' = T_i(\theta)C_i$, $i = 1, \ldots, m$. Since a representation is a homomorphism from rotations, the coefficient $T_i(\theta)$ must satisfy $T_i(\theta')T_i(\theta) = T_i(\theta'+\theta)$. Since a rotation by 2π is the same as no rotation, $T_i(\theta)$ must be a periodic function in θ: $T_i(0) = 1$, $T_i(\theta+2\pi) = T_i(\theta)$. Since $T_i(\theta)$ must be a continuous function, it must have the form $e^{-in\theta}$ for some integer n, where i is the imaginary unit.

Let us call the integer n the *weight* of the irreducible representation.* We call an image characteristic of weight 0 an *absolute invariant* and that of nonzero weight a *relative invariant* of weight n. We also call absolute and relative invariants simply *invariants*.

In sum, any representation (2.1) defined by image characteristics $c_1, \ldots, c_m$ is reduced by taking appropriate linear combinations

$$\begin{bmatrix} C_1 \\ \cdot \\ \cdot \\ \cdot \\ C_m \end{bmatrix} = \begin{bmatrix} & & \\ & P & \\ & & \end{bmatrix} \begin{bmatrix} c_1 \\ \cdot \\ \cdot \\ \cdot \\ c_m \end{bmatrix} \tag{3.1}$$

to the direct sum of one-dimensional irreducible representations of the form

$$\begin{bmatrix} C_1' \\ \cdot \\ \cdot \\ \cdot \\ C_m' \end{bmatrix} = \begin{bmatrix} & & \\ & P & \\ & & \end{bmatrix} \begin{bmatrix} & & \\ & T(\theta) & \\ & & \end{bmatrix} \begin{bmatrix} & & \\ & P^{-1} & \\ & & \end{bmatrix} \begin{bmatrix} C_1 \\ \cdot \\ \cdot \\ \cdot \\ C_m \end{bmatrix}$$

$$= \begin{bmatrix} e^{-in_1\theta} & & & \\ & \cdot & & \\ & & \cdot & \\ & & & e^{-in_m\theta} \end{bmatrix} \begin{bmatrix} C_1 \\ \cdot \\ \cdot \\ \cdot \\ C_m \end{bmatrix}. \tag{3.2}$$

In other words, a single *fixed* non-singular matrix P, which does not depend on θ, can diagonalize. the matrix $T(\theta)$ for *all* values of θ simultaneously. The reason why this is possible is, as is well known, that the two-dimensional rotation group $SO(2)$ is *compact*, hence fully reducible, and also *Abelian*, hence decomposable into one-dimensional irreducible representations due to *Schur's lemma* (cf. Hammermesh[1], Kanatani[8]).

Since each of the invariants defines an irreducible representation, it describes, according to Weyl's thesis, a single particular property of the image.

4. SCALARS, VECTORS AND TENSORS

If an image characteristic c does not change its value if the coordinate system is rotated, i.e.,

$$c' = c, \tag{4.1}$$

it is called a *scalar* (with respect to coordinate rotation). Eqn (4.1) trivially defines a representation, the *identity representation*. Hence, a scalar is an absolute invariant.

A set of two image characteristics a, b is called a *vector* (with respect to coordinate rotation), if it is transformed under coordinate rotation of angle θ counterclockwise by

$$\begin{bmatrix} a' \\ b' \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}. \tag{4.2}$$

This equation defines a *faithful* representation of $SO(2)$, which is called the *vector representation*.

This representation is not irreducible; if we take linear combinations $a+ib$, $a-ib$, we obtain

$$\begin{bmatrix} a'+ib' \\ a'-ib' \end{bmatrix} = \begin{bmatrix} e^{-i\theta} & \\ & e^{i\theta} \end{bmatrix} \begin{bmatrix} a+ib \\ a-ib \end{bmatrix}. \tag{4.3}$$

Thus, $z = a + ib$, $z^* = a - ib$ are relative invariants of weight 1 and - 1 respectively:

$$z' = e^{-i\theta} z, \qquad z^{*\prime} = e^{i\theta} z^*. \tag{4.4}$$

A set of image characteristics A, B, C, D is called a *tensor* (of *rank* 2) with respect to coordinate rotation, if it is transformed by rotation of angle θ counterclockwise in the form

$$\begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}. \tag{4.5}$$

This equation is a linear mapping from A, B, C, D onto A', B', C', D', which is called the *tensor representation*. If we pick out the matrix components, eqn (4.5) is rearranged into the form

$$\begin{bmatrix} A' \\ B' \\ C' \\ D' \end{bmatrix} = \begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta & \cos\theta\sin\theta & \sin^2\theta \\ -\cos\theta\sin\theta & \cos^2\theta & -\sin^2\theta & \cos\theta\sin\theta \\ -\cos\theta\sin\theta & -\sin^2\theta & \cos^2\theta & \cos\theta\sin\theta \\ \sin^2\theta & -\cos\theta\sin\theta & -\cos\theta\sin\theta & \cos^2\theta \end{bmatrix} \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix}. \tag{4.6}$$

This representation is not irreducible. First, note that the matrix of A, B, C, D is uniquely decomposed into its *symmetric part* and *antisymmetric* (or *skewsymmetric*) *part*:

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A & (B+C)/2 \\ (B+C)/2 & D \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & -(C-B)/2 \\ (C-B)/2 & 0 \end{bmatrix}. \tag{4.7}$$

It can be easily checked that this decomposition is *invariant* with respect to coordinate rotation. Namely, the symmetric and antisymmetric parts are transformed independently as tensors. (This is true of tensors of any dimensionality.) Hence, $C - B$ in

* We adopt the convention of putting $e^{-in\theta}$ to define the weight instead of putting $e^{in\theta}$. This is because we consider rotations of the "coordinate system". If we consider rotations of "images" relative to a fixed coordinate system, it is more convenient to put $e^{in\theta}$ to define the weight.

the antisymmetric part is an absolute invariant.

The symmetric part is further decomposed into its *scalar part* (multiple of the unit matrix) and *deviator part* (symmetric matrix of trace 0) uniquely:

$$\begin{bmatrix} A & (B+C)/2 \\ (B+C)/2 & D \end{bmatrix}$$

$$=\frac{A+D}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}+\begin{bmatrix} (A-D)/2 & (B+C)/2 \\ (B+C)/2 & -(A-D)/2 \end{bmatrix}. \qquad (4.8)$$

Again, it can be easily checked that this decomposition is invariant; the two parts are transformed independently. (Again, this is true of symmetric tensors of any dimensionality.) Hence, $(A+B)/2$ of the scalar part must be an absolute invariant. From the deviator part, we can construct relative invariants $(A-B)+i(B+C)$, $(A-D)-i(B+C)$, whose weights are 2 and - 2 respectively.*

From the above consideration, eqn (4.5) is rearranged into the following form:

$$\begin{bmatrix} A'+B' \\ B'-C' \\ (A'-D')+i(B'+C') \\ (A'-D')-i(B'+C') \end{bmatrix}$$

$$=\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & e^{-2i\theta} & \\ & & & e^{2i\theta} \end{bmatrix}\begin{bmatrix} A+B \\ B-C \\ (A-D)+i(B+C) \\ (A-D)-i(B+C) \end{bmatrix}. \qquad (4.9)$$

In other words, if we define new image characteristics by

$$T=A+D, \qquad R=B-C, \qquad S=(A-D)+i(B+C), \qquad (4.10)$$

then T and R are absolute invariants and S is a relative invariant of weight 2:

$$T'=T, \qquad R'=R, \qquad S'=e^{-2i\theta}S. \qquad (4.11)$$

Since they define irreducible representations, each of them should describe, according to Weyl's thesis, a single particular property of the image.

In the above, we derived irreducible representations by decomposing a tensor according to its symmetry and trace. According to a theorem due to Weyl, this is true of tensor representation of any dimension and rank, and *all* irreducible representations of any tensor representation can be obtained systematically.[15]

5. STRUCTURE AND MOTION FROM OPTICAL FLOW

Suppose we take an xyz-coordinate system in the scene and regard the xy-plane as the image plane, choosing $(0, 0, -f)$, the point on the z-axis at distance f from the image plane on the negative side, as the "viewpoint" or "camera focus" (Fig. 1). Let us assume a simple camera model of perspective projection; a point in the scene is projected to the intersection between the image plane and the ray connecting the point and the viewpoint. Then, point (X, Y, Z) in the scene is projected to point (x, y) on the image plane, where

$$x=fX/(f+Z), \qquad y=fY/(f+Z). \qquad (5.1)$$

Orthographic projection is obtained simply by taking the limit of $f \to \infty$.

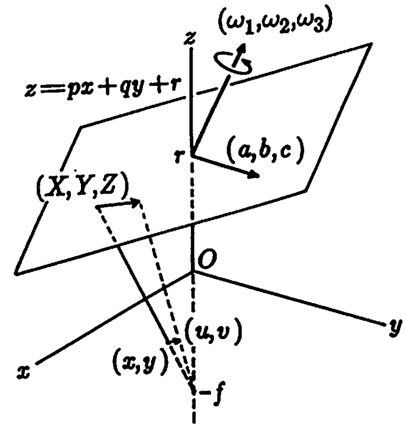

Fig. 1 A plane having equation $z = px + qy + r$ is moving with translational velocity (a, b, c) at $(0, 0, r)$ and rotation velocity $(\omega_1, \omega_2, \omega_3)$ around it. An optical flow is induced on the xy-plane by perspective projection, $(0, 0, -f)$ being the viewpoint.

Suppose a planar surface is moving in the scene, and let $z = px + qy + r$ be its equation. The pair of p, q designates the "gradient" of the plane, and r the distance of the plane from the image plane along the z-axis, which we call the *absolute depth*. An instantaneous rigid motion is specified by the velocity (a, b, c) at a "reference point" (the *translation velocity*) and the *rotation velocity* $(\omega_1, \omega_2, \omega_3)$ around it, i.e., with $(\omega_1, \omega_2, \omega_3)$ as the rotation axis and $\sqrt{\omega_1^2+\omega_2^2+\omega_3^2}$ (rad/sec) as the angular velocity screwwise around it. We choose, as the reference point, $(0, 0, r)$, the intersection of the z-axis with the surface (Fig. 1). Thus, the object parameters to be determined are p, q, r, a, b, c, ω_1, ω_2, ω_3.

If the motion is as described above, the *optical flow* $\dot{x} = u(x, y)$, $\dot{y} = v(x, y)$ induced on the image plane is given as follows (Appendix A):

$$u(x,y)=u_0+Ax+By+(Ex+Fy)x,$$

$$v(x,y)=v_0+Cx+Dy+(Ex+Fy)y. \qquad (5.2)$$

The eight coefficients u_0, v_0, A, B, C, D, E, F, which are the image characteristics in this case, are given as follows (Appendix A).

$$u_0=\frac{fa}{f+r}, \qquad v_0=\frac{fb}{f+r},$$

$$A=p\omega_2-\frac{pa+c}{f+r}, \qquad B=q\omega_2-\omega_3-\frac{qa}{f+r},$$

* If the original image characteristics are real, it can be shown that relative invariants always appear as complex conjugate pairs having weights of opposite signs.

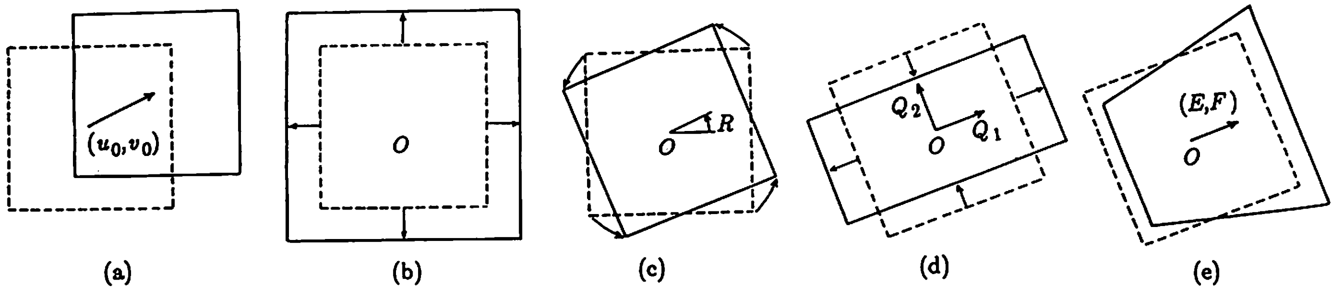

Fig. 2 (a) Translation by (u_0, v_0). (b) Divergence by T. (c) Rotation by R. (d) Shearing with Q_1 and Q_2 as the axes of maximum extension and maximum compression. (e) *Fanning* along (E, F).

$$C=-p\omega_1+\omega_3-\frac{pb}{f+r}, \qquad D=-q\omega_1-\frac{qb+c}{f+r},$$

$$E=\frac{1}{f}(\omega_2+\frac{pc}{f+r}), \qquad F=\frac{1}{f}(-\omega_1+\frac{qc}{f+r}). \tag{5.3}$$

Suppose the image characteristics u_0, v_0, A, B, C, D, E, F are measured from the observed optical flow, say by fitting eqns (5.2) to the flow by the least square method. (Kanatani[2,3], and Kanatani and Chou[10] proposed methods to determine them by measuring *features* of the image at each frame without using the optical flow or the knowledge of the point-to-point correspondence.) Then, eqns (5.3) are regarded as the 3D recovery equations for the unknown object parameters p, q, r, a, b, c, ω_1, ω_2, ω_3. This problem was analyzed and given analytical solutions by Longuet-Higgins,[11] Subbarao and Waxman,[13] and Kanatani.[6] In the following we follow Kanatani.[6]

Applying coordinate rotations to eqns (5.2), we can easily see that u_0, v_0 and E, F are transformed as vectors while A, B, C, D are transformed as a tensor (Appendix B). Hence, we obtain invariants

$$U_0=u_0+iv_0, \qquad T=A+D, \qquad R=C-B,$$

$$S=(A-D)+i(B+C), \qquad K=E+iF. \tag{5.4}$$

where T, R are absolute invariants, U_0, K are relative invariants of weight 1, and S is a relative invariant of weight 2.

Since these invariants define irreducible representations, each of them should have a distinct meaning according to Weyl's thesis. In fact, U_0 represents *translation*, T *divergence*, R *rotation*, S *shearing*, and K what we call *fanning* (or *foreshortening*) of the optical flow as shown in Fig. 2 (Kanatani[6]).

Similarly, we can construct invariant parameters from the object parameters. It is immediately seen that the gradient components p, q, the translation velocities a, b and the rotation velocities ω_1, ω_2 are transformed as vectors, while r, c, ω_3 are scalars (cf. Kanatani[8]). Hence, we can define the following relative invariants of weight 1:.

$$P=p+iq, \qquad V=a+ib, \qquad W=\omega_1+i\omega_2. \tag{5.5}$$

In terms of these invariants, the 3D recovery equations (5.3) are rewritten as

$$V=\frac{f+r}{f}U_0, \qquad PW'^*=(2\omega_3-R)-i(2c'+T),$$

$$PW'=iS, \qquad c'P-iW'=L, \tag{5.6}$$

where we put $c' \equiv c/(f+r)$, $W' \equiv W - iU_0/f$ and $L \equiv fK - U_0/f$. Suppose we know that $c \neq 0$ (Appendix C). Then, the solution is given as follows (Appendix D).

THEOREM 1. The cubic equation

$$X^3+TX^2+\frac{1}{4}(T^2-|S|^2-|L|^2)X$$

$$+\frac{1}{8}(\mathrm{Re}[L^2S]-T|L|^2)=0 \tag{5.7}$$

has three real roots. Let c' be the middle one. Then, the object parameters are given by

$$V=\frac{f+r}{f}U_0, \quad c=(f+r)c', \quad P=\frac{1}{2c'}(L\pm\sqrt{L^2-4c'S}),$$

$$W=\frac{i}{2}(L\mp\sqrt{L^2-4c'S})+\frac{i}{f}U_0,$$

$$\omega_3=\frac{1}{2}R\pm\mathrm{Im}[L^*\sqrt{L^2-4c'S}]. \tag{5.8}$$

(Re and Im denote the real and the imaginary part, respectively, and $*$ designates the complex conjugate.) Thus, (i) the absolute depth r is indeterminate, (ii) $a/(f+r)$, $b/(f+r)$, $c/(f+r)$ are uniquely determined, and (iii) two sets of solutions exist for p, q, ω_1, ω_2, ω_3.

Various related considerations including the "adjacency condition" to check if two planes belong to the same object as well as numerical examples are found in Kanatani.[6]

Note that the weight of the product of two invariants is the sum of their weights, and taking the complex conjugate alters the sign of the weight. In equations expressed in term of invariants like eqns (5.6) - (5.7), *only terms of the same weight can be added or subtracted*. This enables us to check equations very easily and sometimes guess the final form of the solution. This is one of the great advantages of expressing equations in term of invariants.

If we take the limit of $f \to \infty$ in eqns (5.3), we obtain the following *orthographic approximation*:

$$u_0=a, \qquad v_0=b,$$

$$A=p\omega_2,\ B=q\omega_2-\omega_3,\ C=-p\omega_1+\omega_3,\ D=-q\omega_1, \qquad (5.9)$$

In terms of invariants, these 3D recovery equations are rewritten as

$$U_0=V, \quad PW^*=2\omega_3-(R+iT), \quad PW=iS. \qquad (5.10)$$

The first equation gives V. The rest of the equations yields the following analytical solution (Appendix E):

THEOREM 2.

$$\omega_3=\frac{1}{2}(R\pm\sqrt{SS^*-T^2}),$$

$$W=k\exp i\left(\frac{\pi}{4}+\frac{1}{2}\arg(S)-\frac{1}{2}\arg(2\omega_3-(R+iT))\right),$$

$$P=\frac{S}{k}\exp i\left(\frac{\pi}{4}-\frac{1}{2}\arg(S)+\frac{1}{2}\arg(2\omega_3-(R+iT))\right), \qquad (5.11)$$

where k is an indeterminate parameter. For each k, there exist two solutions of p, q, ω_1, ω_2, ω_3.

Other considerations and applications including the adjacency condition are given by Kanatani.[5] Essentially the same problem was also solved by Sugihara and Sugie.[14] Since they did not use invariants, they were unable to obtain analytical solutions and presented only numerical algorithms. Yet, thus computed solutions contain still physically impossible ones, and their algorithms alone cannot reject them. No such solutions exist in our analytical solutions.

If we put $f^2 \to \infty$ but retain terms of $O(f)$, equations for E, F in eqns (5.3) are replaced by

$$E=\omega_2/f, \quad F=-\omega_1/f, \qquad (5.12)$$

respectively. Kanatani called this approximation the *pseudo-orthographic approximation*.[6] Under this approximation, the last of eqns (5.6) is replaced by $K = -iW/f$. Then, the analytical solution is readily oftained as follows:

THEOREM 3.

$$V=\frac{f+r}{f}U_0, \quad P=S/L, \quad W=ifK,$$

$$\omega_3=\frac{1}{2}(R+\mathrm{Im}[Se^{-2i\alpha}]), \quad c=\frac{f+r}{2}(T-\mathrm{Re}[Se^{-2i\alpha}]). \qquad (5.12)$$

Thus, (i) the absolute depth r is indeterminate, (ii) $a/(f+r)$, $b/(f+r)$, $c/(f+r)$ are uniquely determined, and (iii) p, q, ω_1, ω_2, ω_3 are uniquely determined. In particular, *no spurious solution exists*.

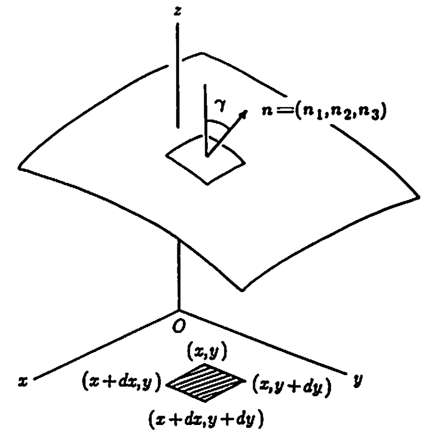

Fig. 3 Under orthographic projection along the z-axis, the area of the region on the surface which corresponds to the infinitesimal square on the xy-plane defined by four points (x, y), $(x + dx, y)$, $(x + dx, y + dy)$, $(x, y + dy)$ is $dxdy/\cos\gamma$, where γ is the angle made by the unit normal **n** to the surface and the z-axis.

The geometrical meaning of this solution is discussed by Kanatani.[6]

If the object is a curved surface, the induced optical flow has the form

$$u(x,y)=u_0+Ax+By+Ex^2+2Fxy+Gy^2+\ldots,$$

$$v(x,y)=v_0+Cx+Dy+Kx^2+2Lxy+My^2+\ldots, \qquad (5.13)$$

where ... denotes higher order terms in x, y (Subbarao.[12] In this case, we obtain the following invariants (Kanatani[8]):

$$U_0=u_0+iv_0, \qquad T=A+D, \qquad R=C-B,$$

$$S=(A-D)+i(B+C), \quad H=(E+2L-G)+i(M+2F-K),$$

$$I=(E-2L+3G)+i(M-2F+3K),$$

$$J=(E-2L-G)-i(M-2F-K). \qquad (5.13)$$

Here, T and R are absolute invariants, U_0, H and I are relative invariants of weight 1, S is a relative invariants of weight 2, and J is a relative invariants of weight 3. Analytical solutions for the shape and motion of a quadric surface can be expressed in term of these invariants (Subbarao[10]).

6. SHAPE FROM TEXTURE FOR CURVED SURFACES

Suppose a textured surface $z = z(x, y)$ is orthographically projected onto the xy-plane along the z-axis (Fig. 3). Assume that the texture is *homogeneous*. (For a precise definition, see Kanatani and Chou[2].) Suppose we can measure the texture density $\Gamma(x, y)$ of the projected image, say by counting the number of texture elements per unit area. (Again, for the exact procedure, see Kanatani and

Chou.[9]) From Fig. 3, we obtain the relationship

$$\Gamma(x,y)=\rho\sqrt{(\partial z/\partial x)^2+(\partial z/\partial y)^2+1}, \qquad (6.1)$$

where ρ is the 'true texture density' of the surface.

Assume, for simplicity, that the surface is quadric:

$$z=r+px+qy+ax^2+2bxy+cy^2. \qquad (6.2)$$

Hence, p, q, r, a, b, c plus the true texture density ρ are the object parameters. From eqns (6.1) and (6.2), we obtain

$$\Gamma(x,y)=A_0\sqrt{1+A_1x+A_2y+A_3x^2+2A_4xy+A_5y^2}, \qquad (6.3)$$

where

$$A_0=\rho\sqrt{1+p^2+q^2}, \quad A_1=\frac{4(ap+bq)}{1+p^2+q^2}, \quad A_2=\frac{4(bp+cq)}{1+p^2+q^2},$$

$$A_3=\frac{4(a^2+b^2)}{1+p^2+q^2}, \quad A_4=\frac{4b(a+c)}{1+p^2+q^2}, \quad A_5=\frac{4(b^2+c^2)}{1+p^2+q^2}. \qquad (6.4)$$

By fitting the form of eqn (6.3) to the observed $\Gamma(x,y)$, we can estimate the parameters A_i, i = 0, ..., 5. (Kanatani and Chou used an indirect method without measuring the texture density.[9]) Hence, they become the image characteristics in this case. Our task is to recover the object parameters ρ, p, q, r, a, b, c from the image characteristics A_i, i = 0, ..., 5. To do this, given the values of the image characteristics A_i, i = 0, ..., 5, we solve eqns (6.4) for unknowns ρ, p, q, r, a, b, c. Hence, eqns (6.4) are the 3D recovery equations.

From the form of eqn (6.3), we can immediately see that A_0 is transformed as a scalar, that A_1, A_2 are transformed as a vector, and that A_3, A_4, A_4, A_5 are transformed as a tensor (cf. Kanatani[8]). Hence, we can define the following invariants:

$$V=\frac{A_1+iA_2}{4}, \quad T=\frac{A_3+A_5}{8}, \quad S=\frac{A_3-A_5}{8}+i\frac{A_4}{4}. \qquad (6.5)$$

Here, T is an absolute invariant, V is a relative invariant of weight 1, and S is a relative invariant of weight 2. (A_0 is itself an absolute invariant.)

Similarly, consider the object parameters. From eqn (6.2), we can conclude that r is transformed as a scalar, that p, q are transformed as a vector and a, b, b, c are transformed as a tensor (cf. Kanatani[8]). Hence, we can define the following invariants:

$$k=\sqrt{1+p^2+q^2}, \quad v=\frac{p+iq}{k}, \quad t=\frac{a+c}{2k}, \quad s=\frac{a-c}{2k}+i\frac{b}{k}. \qquad (6.6)$$

Here, k and t are absolute invariants, v is a relative invariant of weight 1, and s is a relative invariant of weight 2. (ρ is itself an absolute invariant.)

In terms of these invariants, the 3D recovery equations (3.4) are now rewritten as

$$\rho k=A_0, \quad tv+sv^*=V, \quad t^2+ss^*=T, \quad ts=\frac{S}{2}, \qquad (6.7)$$

where ρ, k, v, t, s are unknowns. If we assume $t \neq 0$ and $t^2 - ss^* \neq 0$, the solution is given as follows (Appendix F).

THEOREM 4.

$$t=\pm\sqrt{\frac{T\pm\sqrt{T^2-SS^*}}{2}}, \qquad s=\frac{S}{2t},$$

$$v=\frac{tV-sV^*}{t^2-ss^*}, \quad k=\frac{1}{\sqrt{1-vv^*}}, \quad \rho=\frac{A_0}{k}. \qquad (6.8)$$

Hence, the original object parameters ρ, p, q, a, b, c are given as follows:

$$\rho=A_0/k, \quad p=k\mathrm{Re}[v], \quad q=k\mathrm{Im}[v],$$

$$a=k(t+\mathrm{Re}[s]), \quad b=k\mathrm{Im}[s], \quad c=k(t-\mathrm{Re}[s]). \qquad (6.9)$$

Suppose the surface is not a plane, i.e., parameters a, b, c are not zero at the same time. In the above, we assumed $t \neq 0$. It can be shown that $t = 0$ occurs if the two *principal curvatures* of the surface have the same magnitude, in which case we cannot tell whether the surface is *elliptic* (i.e., the *Gaussian curvature* is positive) or *hyperbolic* (i.e., the Gaussian curvature is negative).

We also assumed $t^2 - ss^* \neq 0$. It can be shown that if $t^2 - ss^* = 0$, or equivalently $ac - b^2 = 0$, the *Hessian* of the surface is zero (and hence the Gaussian curvature is also zero), and the surface is *parabolic*. In this case, only the ratio p:q is determined, indicating the *asymptotic direction* or the 'ridge' of the surface.

Theorem 4 indicates existence of four solutions. This is an essential characteristic of orthographic projection. Firstly, the the projected image is not affected if we take the *mirror image* of the surface with respect to a 'mirror' perpendicular to the z-axis. Hence, the four solutions consist of two pairs of mirror images. The remaining ambiguity occurs because the texture density only tells about the angle of surface inclination (or the *slant*) but nothing about the orientation of inclination (or the *tilt*).

7. CONCLUDING REMARKS

The emphasis in this paper is the importance of choosing good object parameters and image characteristics. Taking examples from optical flow analysis and 'shape from texture', we have demonstrated that a good choice of parameters makes the 3D recovery equations very simple and enables one to obtain analytical solutions.

Our reasoning of choosing parameters is given by group representation theory. We note that the image plane does not have an inherent coordinate system and that one coordinate system and another one obtained by rotating it are equivalent. From this consideration, we constructed invariants which define irreducible representations of the two-dimensional rotation group $SO(2)$. The use of invariants not only makes the 3D recovery equations very simple, enabling us to obtain analytical solutions easily, but also makes the geometrical meaning of the equations very clear, since those invariants correspond to particular properties (Weyl's thesis).

In applying Weyl's thesis, 'the group of transformations which do not essentially change the meaning of the problem' is not limited to the two-dimensional rotation group $SO(2)$ of coordinate rotations. For instance, if the camera is rotated around the center of its lens, the geometry of the rays observed by the camera is essentially identical (if we ignore the existence of image boundaries). Hence, we can define, on the image plane, invariants with respect to image transformations due to camera rotations and obtain invariants corresponding to irreducible representations of the three-dimensional rotation group $SO(3)$.

Since $SO(3)$ is not Abelian, the irreducible representations are not necessarily one-dimensional. However, we can construct invariants by taking not only linear combinations but also general *algebraic* expressions of image characteristics. This idea also leads to many interesting applications to computer vision problems (e.g., Kanatani[4,7,8]). Thus, the principle described here has a wide range of applications to many problems in computer vision and image understanding.

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APPENDIX A

If a point (X, Y, Z) in the scene is on the plane $z = px + qy + r$, there is a one-to-one correspondence between the point (X, Y, Z) in the scene and its projection (x, y) on the image plane. In fact, solving eqns (5.1) and $Z = pX + qY + r$ simultaneously for X, Y, Z, we obtain

$$X=\frac{(f+r)x}{f-px-qy},\quad Y=\frac{(f+r)y}{f-px-qy},\quad Z=\frac{f(px+qy+r)}{f-px-qy}. \qquad (A.1)$$

The velocity of point (X, Y, Z) in the scene is given by

$$\begin{bmatrix} \dot{X} \\ \dot{Y} \\ \dot{Z} \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} + \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} \times \begin{bmatrix} X \\ Y \\ Z-r \end{bmatrix}. \qquad (A.2)$$

Substituting $Z = pX + qY + r$, we obtain

$$\dot{X}=a+p\omega_2X+(q\omega_2-\omega_3)Y, \quad \dot{Y}=b+(\omega_3-p\omega_1)X-q\omega_1Y,$$
$$\dot{Z}=c-\omega_2X+\omega_1Y. \qquad (A.3)$$

Differentiating both sides of eqns (5.1), we obtain the velocity of the image point (x,y) as follows:

$$\dot{x}=\frac{f\dot{X}}{f+Z}-\frac{x\dot{Z}}{f+Z}, \quad \dot{y}=\frac{f\dot{Y}}{f+Z}-\frac{y\dot{Z}}{f+Z}. \qquad (A.4)$$

From eqns (A.3) and eqns (5.1), we see that

$$\frac{f\dot{X}}{f+Z}=\frac{fa}{f+Z}+p\omega_2x+(q\omega_2-\omega_3)y, \quad \frac{f\dot{Y}}{f+Z}=\frac{fb}{f+Z}+(\omega_3-p\omega_1)x-q\omega_1y,$$

$$\frac{fZ}{f+Z}=\frac{fc}{f+Z}-\omega_2 x+\omega_1 y. \quad (A.5)$$

Substituting these in eqns (A.5) and $1/(f+Z)$ = $(f-px-qy)/f(f+r)$, which is obtained from the last of eqns (A.2), we obtain the result of eqns (5.2) and (5.3).

APPENDIX B

Let us define vectors

$$\boldsymbol{u}=\begin{bmatrix} u \\ v \end{bmatrix},\ \boldsymbol{u}_0=\begin{bmatrix} u_0 \\ v_0 \end{bmatrix},\ \boldsymbol{x}=\begin{bmatrix} x \\ y \end{bmatrix},\ \boldsymbol{k}=\begin{bmatrix} E \\ F \end{bmatrix}, \quad (B.1)$$

and matrices

$$\boldsymbol{A}=\begin{bmatrix} A & B \\ C & D \end{bmatrix},\quad \boldsymbol{R}(\theta)=\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}. \quad (B.2)$$

The optical flow of eqns (5.2) is expressed as

$$\boldsymbol{u}=\boldsymbol{u}_0+\boldsymbol{A}\boldsymbol{x}+(\boldsymbol{k},\boldsymbol{x})\boldsymbol{x}, \quad (B.3)$$

where $(\cdot,\cdot)$ denotes the inner product.

Suppose we use an $x'y'$-coordinate system obtained by rotating the xy-coordinate system around the z-axis by angle θ counterclockwise. Since we are observing the rigid motion of a plane, the optical flow *must* have the same form

$$u'=u_0'+A'x'+B'y'+(E'x'+F'y')x',$$

$$v'=v_0'+C'x'+D'y'+(E'x'+F'y')y', \quad (B.4)$$

or

$$\boldsymbol{u}'=\boldsymbol{u}_0'+\boldsymbol{A}'\boldsymbol{x}'+(\boldsymbol{k}',\boldsymbol{x}')\boldsymbol{x}'. \quad (B.5)$$

We know that the coordinates x, y and the velocity components u, v are transformed under coordinate rotation as vectors:

$$\boldsymbol{x}'=\boldsymbol{R}(\theta)\boldsymbol{x},\quad \boldsymbol{u}'=\boldsymbol{R}(\theta)\boldsymbol{u}. \quad (B.6)$$

Hence, from eqn (B.3), we obtain

$$\boldsymbol{u}'=\boldsymbol{R}(\theta)(\boldsymbol{u}_0+\boldsymbol{A}\boldsymbol{x}+(\boldsymbol{k},\boldsymbol{x})\boldsymbol{x})$$

$$=\boldsymbol{R}(\theta)(\boldsymbol{u}_0+\boldsymbol{A}\boldsymbol{R}(\theta)^T\boldsymbol{x}'+(\boldsymbol{k},\boldsymbol{R}(\theta)^T\boldsymbol{x}')\boldsymbol{R}(\theta)^T\boldsymbol{x}')$$

$$=\boldsymbol{R}(\theta)\boldsymbol{u}_0+\boldsymbol{R}(\theta)\boldsymbol{A}\boldsymbol{R}(\theta)^T\boldsymbol{x}'+(\boldsymbol{R}(\theta)\boldsymbol{k},\boldsymbol{x}')\boldsymbol{x}'. \quad (B.7)$$

Note that $\boldsymbol{R}(\theta)^{-1}$ = $\boldsymbol{R}(\theta)^T$ and $(\boldsymbol{R}(\theta)\cdot,\cdot)$ = $(\cdot,\boldsymbol{R}(\theta)^T\cdot)$. Comparing this with eqn (B.5), we find

$$\boldsymbol{u}_0'=\boldsymbol{R}(\theta)\boldsymbol{u}_0,\quad \boldsymbol{A}'=\boldsymbol{R}(\theta)\boldsymbol{A}\boldsymbol{R}(\theta)\boldsymbol{k}. \quad (B.8)$$

Hence, $\boldsymbol{u}_0$ and $\boldsymbol{k}$ are transformed as vectors, while $\boldsymbol{A}$ is transformed as a tensor.

APPENDIX C

LEMMA C.1 We have c = 0 if and only if

$$\mathrm{Re}\,[Se^{-2i\alpha}]=T \quad (C.1)$$

is satisfied (within a certain threshold), where α $\equiv$ $\arg(L)$. Then, the solution of eqns (5.6) is given by

$$V=\frac{f+r}{f}U_0,\ P=\frac{S}{L},\ W=ifK,\ \omega_3=\frac{1}{2}(R+\mathrm{Im}\,[Se^{-2i\alpha}]). \quad (C.2)$$

Proof. If c = 0, we can immediately obtain eqns (C.2). The only remaining condition to be satisfied is $\mathrm{Im}\,[PW'^*]$ = $-iT$. From eqns (C.2), we see that

$$PW'^*=iS\frac{fK^*-U_0^*/f}{fK-U_0/f}=-iSe^{-2i\alpha}. \quad (C.3)$$

Hence, we obtain eqn (C.1).

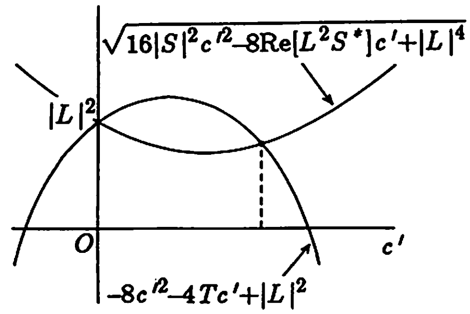

Fig. 4 Existence and uniqueness of nonzero c'.

APPENDIX D

If $c'\neq 0$, the third of eqns (5.6) is rewritten as $(c'P)(-iW')$ = $c'S$. Hence, the last two of eqns (5.6) imply that $c'P$ and $-iW'$ are the two roots of the quadratic equation

$$X^2-LX+c'S=0, \quad (D.1)$$

where L is defined as indicated in Theorem 1. Hence, P and W' are given as functions of c' by

$$P(c')=\frac{1}{2c'}(L\pm\sqrt{L^2-4c'S}),\ c'=\frac{i}{2}(L\mp\sqrt{L^2-4c'S}). \quad (D.2)$$

Taking the real and the imaginary parts of the second of eqns (5.6), we obtain

$$\omega_3=\frac{1}{2}(R+\mathrm{Re}\,[P(c')W'(c')^*]),$$

$$c'=-\frac{1}{2}(T+\mathrm{Im}\,[P(c')W'(c')^*]). \quad (D.3)$$

The last of eqns (6.5) is obtained by substituting eqns (D.2) in the first of eqns (D.3). The equation to determine c' is obtained by substituting eqns (D.2) in the second of eqns (D.3) as follows:

$$\sqrt{16|S|^2c'^2-8\mathrm{Re}\,[L^2S^*]\,c'+|L|^4}=8c'^2-4Tc'+|L|^2. \quad (D.4)$$

The left-hand side of eqn (D.4) is a smooth concave function (or constant if S = 0) passing through $(0,\ |L|^2)$ (Fig. 4). Since we know that $c'\neq 0$, we can see from Fig. 4 that there exists a single unique nonzero solution c'. Taking the squares of both sides and dropping off c' from both sides, we obtain eqn (5.7). From Fig. 4, we can easily see that eqn (5.6) has three real roots and that the middle one is the desired root. The other two roots were introduced by squaring of both sides.

APPENDIX E

Since $|PW^*|=|PW|$, the right-hand sides of the last two of eqns (5.10) have the same modulus, i.e.,

$$(2\omega_3-(R+iT))(2\omega_3-(R-iT))=SS^*, \quad (E.1)$$

which gives the first of eqns (5.11). Now, we can see from the last tow of eqns (5.10) that if P and W satisfy them, so do W multiplied by a real number and P divided by that number. Consequently, the magnitude $k = |W|$ of W can be taken as an indeterminate scale factor. Elimination of P from the last two of eqns (5.10) by taking the ratio yields

$$W/W^*=iS/(2\omega_3-(R-iT)). \quad (E.2)$$

Taking the argument of both sides, we obtain

$$2\arg(W)=\frac{\pi}{2}+\arg(S)-\arg(2\omega_3-(R+iT)) \bmod 2\pi, \quad (E.3)$$

and hence

$$\arg(W)=\frac{\pi}{4}+\frac{1}{2}\arg(S)-\frac{1}{2}\arg(2\omega_3-(R+iT)) \bmod \pi. \quad (E.4)$$

However, 'mod π' can be ignored by allowing the scale factor k to be negative. Then, W is obtained in the form of $ke^{i\arg(W)}$, and from the third of eqns (5.10) $P = iS/W$.

APPENDIX F

From the last of eqns (6.7), we obtain the second of eqns (6.8). Substituting it in the third, we obtain

$$t^4-Tt^2+\frac{1}{4}SS^*=0, \quad (F.1)$$

and consequently the first of eqns (6.8). Taking the complex conjugate of the second of eqns (6.7), we obtain $tv^* + s^*v = V^*$. Eliminating v^*, we obtain the third of eqns (6.8). Next, noting

$$1-vv^*=1-\frac{p^2}{1+p^2+q^2}-\frac{q^2}{1+p^2+q^2}=\frac{1}{1+p^2+q^2}=\frac{1}{k^2}, \quad (F.2)$$

we obtain the fourth of eqns (6.8). Finally, from the first of eqns (6.7), we obtain the last of eqns (6.8).