

Renormalization for Unbiased Estimation

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Abstract

In many computer vision problems, one needs to robustly estimate parameter values from a large number of image data. In practice, least-squares minimization is computationally the most convenient. We point out that the least-squares solution is in general "statistically biased" in the presence of noise, and present a scheme called "renormalization", which iteratively removes the statistical bias by automatically adjusting to the image noise. It is applied to estimation of vanishing points and focuses of expansion and conic fitting.

1 Introduction

In many computer vision problems, one needs to robustly estimate parameter values from a large number of image data. Typical examples are estimation of vanishing points and focuses of expansion as common intersections of lines fitted to edges and conics fitted to pixel data.

In such problems, least-squares minimization is computationally the most convenient and practical. However, the least-squares solution is in general "statistically biased" in the presence of noise. Employing a statistical model of noise [10, 11, 12], we analyze the statistical bias in quantitative terms for the problems mentioned above, and present a scheme called "renormalization", which iteratively removes the statistical bias by automatically adjusting to the image noise. Its effectiveness is demonstrated by real image experiments and random number simulations. The speed of convergence of renormalization is proved to be "quadratic".

2 Statistical Model of Noise

Assume the camera imaging model shown in (Fig. 1(a)) [6, 7, 11]. We call the coordinate origin O the *viewpoint* and the constant f the *focal length*.

A point on the image plane is represented by the unit vector \mathbf{m} indicating the orientation of the ray starting from the viewpoint O and passing through that point; a line on the image plane is represented by the unit surface normal \mathbf{n} to the plane passing through the viewpoint O and intersecting the image plane along that line (Fig. 1(a)). We call \mathbf{m} and \mathbf{n} the *N-vectors* of the point and the line [7, 11].

If \mathbf{m} and \mathbf{n} are the N-vectors of a point P and a line l , respectively, point P is on line l , or line l passes

through point P , if and only if

$$(\mathbf{m}, \mathbf{n}) = 0, \quad (1)$$

where (\cdot, \cdot) designates the inner product of vectors. If eq. (1) holds, point P and line l are said to be *incident* to each other [7, 11]. We call eq. (1) the *incidence equation*.

Point and line data detected by applying image operations to real images are not accurate, and N-vectors computed from them are not exact. In this paper, we regard such errors in N-vectors as caused by "noise". Let $\bar{\mathbf{m}}$ be the N-vector of a point on the image plane when there is no noise. In the presence of noise, a perturbed N-vector $\mathbf{m} = \bar{\mathbf{m}} + \Delta\mathbf{m}$ is observed. We regard $\Delta\mathbf{m}$ as a random variable and consider the *covariance matrix*

$$V[\mathbf{m}] = E[\Delta\mathbf{m}\Delta\mathbf{m}^T], \quad (2)$$

where T denotes transpose and $E[\cdot]$ denotes the expectation.

We adopt the following model of image noise: "Noise (in our sense) occurs at each point on the image plane and is equally likely in all orientations with the same root-mean-square ϵ ." Let us call ϵ (measured in pixels) the *image accuracy*. Then, we can obtain the following expression, where I is the unit matrix [10, 11, 12]:

Proposition 1 *If the size of the image is small compared with the focal length f , the covariance matrix $V[\mathbf{m}]$ of the N-vector of a data pixel has the form*

$$V[\mathbf{m}] = \frac{\tilde{\epsilon}^2}{2}(I - \mathbf{m}\mathbf{m}^T), \quad \tilde{\epsilon} = \frac{\epsilon}{f}. \quad (3)$$

3 Intersection Estimation

3.1 Optimal least-squares estimation

Lines meeting at a common intersection are said to be *concurrent*. Let $\{\mathbf{n}_\alpha\}$, $\alpha = 1, \dots, N$, be the N-vectors of concurrent lines $\{l_\alpha\}$. If \mathbf{m} is the N-vector of their common intersection, the incidence equations $(\mathbf{m}, \mathbf{n}_\alpha) = 0$, $\alpha = 1, \dots, N$, hold. Hence, \mathbf{m} is robustly computed from $\{\mathbf{n}_\alpha\}$ by the least-squares optimization

$$J = \sum_{\alpha=1}^N W_\alpha (\mathbf{m}, \mathbf{n}_\alpha)^2 \rightarrow \min, \quad (4)$$

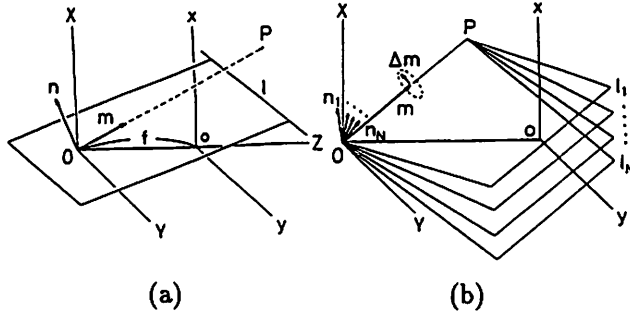


Figure 1: (a) Camera imaging geometry and N-vectors of a point and a line. (b) The common intersection of concurrent lines.

where W_α are nonnegative weights (Fig. 1(b)). The weights W_α should be chosen so that reliable data are given large weights while unreliable data are given small weights. If the noise in the data is independent and if the distribution is approximated by a Gaussian distribution, it can be shown [10, 11, 12] that the optimal weights in the sense of maximum likelihood estimation are given by

$$W_\alpha = \frac{1}{(\mathbf{m}, V[\mathbf{n}_\alpha] \mathbf{m})}. \quad (5)$$

Since the squared sum

$$J = \sum_{\alpha=1}^N W_\alpha (\mathbf{m}, \mathbf{n}_\alpha)^2 = (\mathbf{m}, \left(\sum_{\alpha=1}^N W_\alpha \mathbf{n}_\alpha \mathbf{n}_\alpha^T \right) \mathbf{m}) \quad (6)$$

is a quadratic form in unit vector \mathbf{m} , it is minimized by the unit eigenvector of the *moment matrix*

$$\mathbf{N} = \sum_{\alpha=1}^N W_\alpha \mathbf{n}_\alpha \mathbf{n}_\alpha^T \quad (7)$$

for the smallest eigenvalue.

3.2 Statistical bias

An estimate is *statistically unbiased* if the expectation of the error is zero, and *statistically biased* otherwise. If each N-vector \mathbf{m}_α is perturbed by noise by $\Delta \mathbf{m}_\alpha$, the moment matrix \mathbf{N} of eq. (7) is perturbed accordingly. If we write $\mathbf{N} = \bar{\mathbf{N}} + \Delta \mathbf{N}$, where $\bar{\mathbf{N}}$ is the unperturbed moment matrix, it can be shown [11, 12] that

$$E[\Delta \mathbf{N}] = \sum_{\alpha=1}^N W_\alpha V[\mathbf{n}_\alpha]. \quad (8)$$

Let $\mathbf{m} = \bar{\mathbf{m}} + \Delta \mathbf{m}$ be the unit eigenvector of the perturbed \mathbf{N} for the smallest eigenvalue, where $\bar{\mathbf{m}}$ is the exact solution. According to the *perturbation theorem* (see [11]), the expectation of the perturbation $\Delta \mathbf{m}$ is

$E[\Delta \mathbf{m}] = O(E[\Delta \mathbf{N}])$. Hence, the estimate is in general statistically biased.

Eq. (8) implies that if we define, instead of eq. (7),

$$\hat{\mathbf{N}} = \sum_{\alpha=1}^N W_\alpha (\mathbf{n}_\alpha \mathbf{n}_\alpha^T - V[\mathbf{n}_\alpha]), \quad (9)$$

then $E[\hat{\mathbf{N}}] = \bar{\mathbf{N}}$, and hence the unit eigenvector \mathbf{m} of its smallest eigenvalue is an unbiased estimate of $\bar{\mathbf{m}}$.

However, two problems arise if we are to compute this. Firstly, the optimal weights given by eq. (5) involve the N-vector \mathbf{m} that we want to compute. Secondly, the computation of the matrix $\hat{\mathbf{N}}$ requires the knowledge of the covariance matrices $V[\mathbf{n}_\alpha]$. They involve image noise characteristics, which are difficult to predict a priori.

3.3 Renormalization

The first problem can be solved by iterations: we compute an estimate \mathbf{m} and compute approximately optimal weights W_α , then compute a better estimate of \mathbf{m} , and so on. For the second problem, we take advantage of the fact that, as we will show shortly, the covariance matrix $V[\mathbf{n}_\alpha]$ can be expressed in the form

$$V[\mathbf{n}_\alpha] = c V_0[\mathbf{n}_\alpha], \quad (10)$$

where c is an unknown constant characterizing the magnitude of image noise and $V_0[\mathbf{n}_\alpha]$ has a known form.

Since multiplication of the optimal weights by a constant does not affect the resulting solution, eq. (5) can be replaced by

$$W_\alpha = \frac{1}{(\mathbf{m}, V_0[\mathbf{n}_\alpha] \mathbf{m})}. \quad (11)$$

Ideally, the constant c should be chosen so that $E[\hat{\mathbf{N}}] = \bar{\mathbf{N}}$, but this is impossible unless image noise characteristics are known. On the other hand, if $E[\hat{\mathbf{N}}] = \bar{\mathbf{N}}$, we have

$$E[(\bar{\mathbf{m}}, \hat{\mathbf{N}} \bar{\mathbf{m}})] = (\bar{\mathbf{m}}, E[\hat{\mathbf{N}}] \bar{\mathbf{m}}) = (\bar{\mathbf{m}}, \bar{\mathbf{N}} \bar{\mathbf{m}}) = 0, \quad (12)$$

because from eq. (4) $J = (\mathbf{m}, \mathbf{N} \mathbf{m})$ takes its absolute minimum 0 for the exact solution $\bar{\mathbf{m}}$ in the absence of noise. This suggests that we require that $(\mathbf{m}, \hat{\mathbf{N}} \mathbf{m}) = 0$ at each iteration step. If $(\mathbf{m}, \hat{\mathbf{N}} \mathbf{m}) \neq 0$ for the current estimates c and \mathbf{m} , we define

$$\hat{\mathbf{N}}' = \hat{\mathbf{N}} - \frac{(\mathbf{m}, \hat{\mathbf{N}} \mathbf{m})}{\sum_{\beta=1}^N W_\beta (\mathbf{m}, V_0[\mathbf{n}_\beta] \mathbf{m})} \sum_{\alpha=1}^N W_\alpha V_0[\mathbf{n}_\alpha]. \quad (13)$$

Then, $(\mathbf{m}, \hat{\mathbf{N}}' \mathbf{m}) = 0$. Note that $(\mathbf{m}, \hat{\mathbf{N}} \mathbf{m})$ equals the smallest eigenvalue of $\hat{\mathbf{N}}$. From this observation, we obtain the following procedure, which we call *renormalization* [11, 12, 14]:

renormalization ($\{\mathbf{n}_\alpha\}, \{V_0[\mathbf{n}_\alpha]\}$)

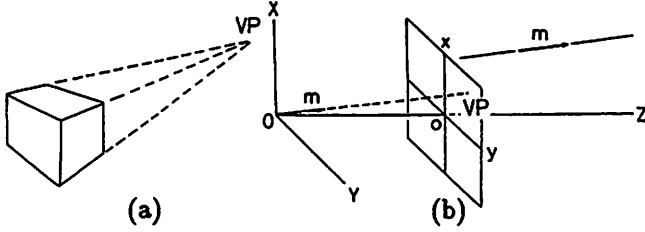


Figure 2: (a) Vanishing point. (b) The N-vector \mathbf{m} of the vanishing point indicates the 3-D orientation of the line.

1. Let $c = 0$ and $W_\alpha = 1$, $\alpha = 1, \dots, N$.
2. Compute the unit eigenvector \mathbf{m} of

$$\hat{N} = \sum_{\alpha=1}^N W_\alpha (\mathbf{n}_\alpha \mathbf{n}_\alpha^\top - c V_0[\mathbf{n}_\alpha]) \quad (14)$$

for the smallest eigenvalue, and let λ_m be the smallest eigenvalue.

3. Update c and W_α as follows:

$$c \leftarrow c + \frac{\lambda_m}{\sum_{\alpha=1}^N W_\alpha (\mathbf{m}, V_0[\mathbf{n}_\alpha] \mathbf{m})}, \quad (15)$$

$$W_\alpha \leftarrow \frac{1}{(\mathbf{m}, V_0[\mathbf{n}_\alpha] \mathbf{m})}. \quad (16)$$

4. Return \mathbf{m} if the update has converged; else go back to Step 2.

3.4 Estimation of vanishing points

If lines $\{l_\alpha\}$, $\alpha = 1, \dots, N$, on the image plane are projections of parallel lines in the scene, they are concurrent on the image plane; their common intersection is their *vanishing point* (Fig. 2(a)). It is easily seen from Fig. 2(b) that the N-vector \mathbf{m} of the vanishing point indicates the 3-D orientation of the corresponding lines in the scene [7]. This means that if lines are fitted, say by least square, to the edge segments resulting from projection of such parallel lines, their 3-D orientation can be computed as the N-vector of their common intersection. Evidently, the reliability of the computed vanishing point depends on the reliability of the lines fitted to the edges, and this problem has been studied by a number of researchers [3, 4, 8, 13, 21].

It can be shown [9, 11, 12] that the covariance matrix $V[\mathbf{n}]$ of the N-vector \mathbf{n} of the line fitted to an edge segment by least square has the form

$$V[\mathbf{n}] \approx \frac{6\kappa}{w^3} \mathbf{u} \mathbf{u}^\top + \frac{\kappa}{2f^2 w} \mathbf{m}_G \mathbf{m}_G^\top, \quad \kappa = \frac{\epsilon^2}{\gamma}. \quad (17)$$

Here, \mathbf{u} is the unit vector indicating the orientation of the edge segment (Fig. 3(a)); \mathbf{m}_G is the N-vector of its center point; w is its length (measured in pixels);

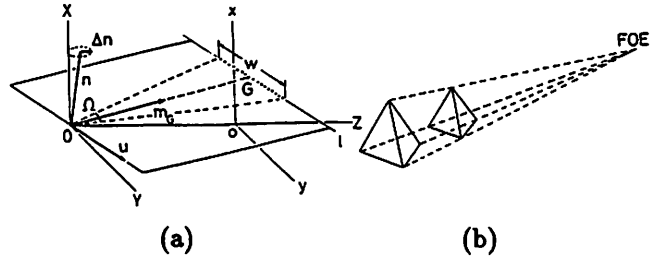


Figure 3: (a) Line fitting to an edge segment. (b) Focus of expansion.

γ is the density of the edge points (i.e., the number of edge points per unit pixel length). The vectors \mathbf{u} and \mathbf{m}_G are formally defined by $\mathbf{u} = \pm N[\mathbf{m}_a - \mathbf{m}_b]$ and $\mathbf{m}_G = \pm N[\mathbf{m}_a + \mathbf{m}_b]$, where \mathbf{m}_a and \mathbf{m}_b are the N-vectors of the end points of the edge segment.

Thus, we can apply renormalization by putting

$$V_0[\mathbf{n}_\alpha] = \frac{6}{w_\alpha^3} \mathbf{u}_\alpha \mathbf{u}_\alpha^\top + \frac{1}{2f^2 w_\alpha} \mathbf{m}_{G\alpha} \mathbf{m}_{G\alpha}^\top, \quad (18)$$

where the subscript α refers to the α th edge segment.

3.5 Estimation of focuses of expansion

Consider two images of a moving object or a stationary scene taken from a moving camera before and after the motion. Identifying the two image frames, let us call the lines joining corresponding points in the two frames *trajectories*. It can be easily deduced from the geometric interpretation of vanishing points (Fig. 2(b)) that if the (object or camera) motion is a pure 3-D translation, all trajectories meet, when extended, at a common intersection called the *focus of expansion* (Fig. 3(b)), and its N-vector indicates the 3-D orientation of the (object or camera) motion [7, 11]. Evidently, the reliability of the focus of expansion depends on the reliability of the trajectories [20].

Let \mathbf{m} and \mathbf{m}' be the N-vectors of two corresponding points on the image plane, and \mathbf{n} the N-vector of the trajectory passing through them. It can be shown [11, 12] that the covariance matrix $V[\mathbf{n}]$ of the N-vector \mathbf{n} is given by

$$V[\mathbf{n}] = \frac{\epsilon^2}{2} \left(\frac{\mathbf{u} \mathbf{u}^\top}{1 - (\mathbf{m}, \mathbf{m}')} + \frac{\mathbf{m}_G \mathbf{m}_G^\top}{1 + (\mathbf{m}, \mathbf{m}')} \right), \quad (19)$$

where \mathbf{u} is the vector that indicates the orientation of the trajectory, while \mathbf{m}_G is the N-vector of the center of the two points (Fig. 4(a)) formally defined by $\mathbf{u} = \pm N[\mathbf{m}' - \mathbf{m}]$ and $\mathbf{m}_G = \pm N[\mathbf{m} + \mathbf{m}']$.

Thus, we can apply renormalization by putting

$$V_0[\mathbf{n}_\alpha] = \frac{1}{2} \left(\frac{\mathbf{u}_\alpha \mathbf{u}_\alpha^\top}{1 - (\mathbf{m}_\alpha, \mathbf{m}'_\alpha)} + \frac{\mathbf{m}_{G\alpha} \mathbf{m}_{G\alpha}^\top}{1 + (\mathbf{m}_\alpha, \mathbf{m}'_\alpha)} \right), \quad (20)$$

where the subscript α refers to the α th trajectory.

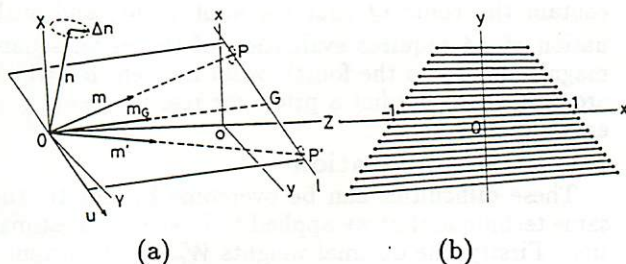


Figure 4: (a) The orientation u and the center point G of a pair of image points. (b) Simulated trajectories.

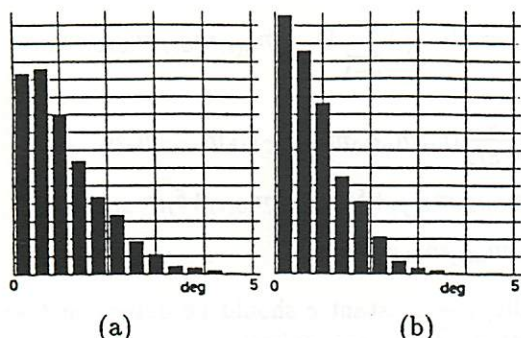


Figure 5: Error in the focus of expansion without renormalization (a) and with renormalization (b).

Example. Fig. 4(b) shows simulated trajectories. The focal length is set to $f = 20$. To the x - and y -coordinates of each end point is added noise obeying an independent normal distribution of mean 0 and standard deviation 0.005. The common intersection is computed 1000 times, each time using different noise. Fig. 5(a) is the histogram of the error (in degrees) in the orientation of the N-vector m of the computed focus of expansion without renormalization. Fig. 5(b) is the histogram obtained by applying renormalization. The computation converges after three or four iterations. It can be clearly seen that accuracy has improved without using any knowledge about the image noise.

4 Conic Fitting

4.1 Optimal least-squares fitting

A conic is a quadratic curve in the form

$$Ax^2 + 2Bxy + Cy^2 + 2(Dx + Ey) + F = 0. \quad (21)$$

If we define the matrix

$$Q = \begin{pmatrix} A & B & D/f \\ B & C & E/f \\ D/f & E/f & F/f^2 \end{pmatrix}, \quad (22)$$

eq. (21) is written in terms of the N-vector m in the form

$$(m, Qm) = 0. \quad (23)$$

For brevity, we hereafter call the conic represented by matrix Q simply "conic Q ". Since a multiple of Q by a constant defines the same image curve, we can adopt the normalization $\|Q\| = 1$, where $\|\cdot\|$ denotes the matrix norm: $\|Q\|^2 = \sum_{i,j=1}^3 Q_{ij}^2$.

It can be shown [5, 11, 15, 16] that the position and orientation of a conic in the scene can be computed analytically from its projection. In order to do such analysis, image conics must be mathematically represented by curve fitting [1, 2, 17, 18, 19]. However, not much has been studied about the statistical error behavior, with the exception of Porrill [17], who applied the Kalman filter in a modified form (the Kalman filter will be discussed later).

Let $\{P_\alpha\}$, $\alpha = 1, \dots, N$, be the pixels to which a conic is to be fitted. Let $\{m_\alpha\}$ be their N-vectors. Our task is to find a conic Q such that ideally $(m_\alpha, Qm_\alpha) = 0$, $\alpha = 1, \dots, N$. Consider the least-squares optimization

$$J(Q) = \sum_{\alpha=1}^N W_\alpha (m_\alpha, Qm_\alpha)^2 \rightarrow \min, \quad (24)$$

where W_α is the weight for the α th pixel. Define the moment tensor $M = (M_{ijkl})$ by

$$M_{ijkl} = \sum_{\alpha=1}^N W_\alpha m_{\alpha(i)} m_{\alpha(j)} m_{\alpha(k)} m_{\alpha(l)}, \quad (25)$$

where $m_{\alpha(i)}$ is the i th component of vector m_α . Tensor $M = (M_{ijkl})$ defines a linear mapping from a matrix to a matrix: MQ is the matrix whose (ij) element is $\sum_{k,l=1}^3 M_{ijkl} Q_{kl}$. If the matrix inner product is defined by $(A, B) = \sum_{i,j=1}^3 A_{ij} B_{ij}$, the problem is written as

$$J(Q) = (Q, MQ) \rightarrow \min, \quad \|Q\| = 1. \quad (26)$$

As for minimizing a quadratic form in a unit vector, the minimum of (26) is attained by the "unit eigenmatrix Q " of "tensor M " for the smallest eigenvalue, where we mean by "a unit eigenmatrix of tensor M for eigenvalue λ " a matrix Q of norm 1 ($\|Q\| = 1$) such that $MQ = \lambda Q$.

The weights W_α must be chosen so that reliable data are given larger weights while unreliable data are given small weight. As in the case of intersection detection, we can show [11, 14] that the optimal weights are given by

$$W_\alpha = \frac{\text{const.}}{\|Qm_\alpha\|^2}. \quad (27)$$

We choose the constant in such a way that $\sum_{\alpha=1}^N W_\alpha = 1$.

4.2 Statistical bias

Let $\bar{\mathcal{M}} = (\bar{M}_{ijkl})$ be the unperturbed moment tensor of exact N-vectors $\bar{\mathbf{m}}_\alpha$. If each N-vector is perturbed into $\mathbf{m}_\alpha = \bar{\mathbf{m}}_\alpha + \Delta\mathbf{m}_\alpha$, the moment tensor \mathcal{M} is perturbed into $\mathcal{M} = \bar{\mathcal{M}} + \Delta\mathcal{M}$ accordingly. It can be shown [11, 14] that the expectation of $\Delta\mathcal{M} = (\Delta M_{ijkl})$ is

$$E[\Delta M_{ijkl}] = \sum_{\alpha=1}^N W_\alpha \left(-3(\bar{\epsilon}^2 - \frac{\bar{\mu}^4}{8}) \bar{m}_{\alpha(i)} \bar{m}_{\alpha(j)} \bar{m}_{\alpha(k)} \bar{m}_{\alpha(l)} + \frac{1}{2}(\bar{\epsilon}^2 - \frac{\bar{\mu}^4}{4}) (\delta_{ij} \bar{m}_{\alpha(k)} \bar{m}_{\alpha(l)} + \delta_{ik} \bar{m}_{\alpha(j)} \bar{m}_{\alpha(l)} + \delta_{il} \bar{m}_{\alpha(k)} \bar{m}_{\alpha(j)} + \delta_{jk} \bar{m}_{\alpha(i)} \bar{m}_{\alpha(l)} + \delta_{jl} \bar{m}_{\alpha(i)} \bar{m}_{\alpha(k)} + \delta_{kl} \bar{m}_{\alpha(i)} \bar{m}_{\alpha(j)}) + \frac{\bar{\mu}^4}{8} (\delta_{ij} \delta_{kl} + \delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) \right), \quad (28)$$

where δ_{ij} is the Kronecker delta and we put $\bar{\epsilon} = \sqrt{E[\|\Delta\mathbf{m}_\alpha\|^2]}$ and $\bar{\mu} = \sqrt[4]{E[\|\Delta\mathbf{m}_\alpha\|^4]}$. According to the perturbation theorem, if $\mathbf{Q} = \bar{\mathbf{Q}} + \Delta\mathbf{Q}$ is the unit eigenmatrix of $\mathcal{M} = \bar{\mathcal{M}} + \Delta\mathcal{M}$, where $\bar{\mathbf{Q}}$ is the exact solution, we have $E[\Delta\mathbf{Q}] = O(E[\Delta\mathcal{M}]) = O(\bar{\epsilon}^2)$. Thus, the solution is statistically biased (the explicit expression of $E[\Delta\mathbf{Q}]$ is given in [11]).

It appears that an unbiased estimate can be obtained by subtracting $E[\Delta\mathcal{M}]$ from \mathcal{M} . However, the expression (28) is given in terms of the "exact" N-vectors $\bar{\mathbf{m}}_\alpha$, and replacing them by \mathbf{m}_α would introduce another source of bias. It can be shown [11, 14] that if we define, instead of eq. (25), a tensor $\hat{\mathcal{M}} = (\hat{M}_{ijkl})$ by

$$\hat{M}_{ijkl} = \sum_{\alpha=1}^N W_\alpha \left((1 - \frac{\bar{\epsilon}^2}{2}) m_{\alpha(i)} m_{\alpha(j)} m_{\alpha(k)} m_{\alpha(l)} - \frac{1}{2}(\bar{\epsilon}^2 - \frac{\bar{\mu}^4}{4}) (\delta_{ij} m_{\alpha(k)} m_{\alpha(l)} + \delta_{ik} m_{\alpha(j)} m_{\alpha(l)} + \delta_{il} m_{\alpha(k)} m_{\alpha(j)} + \delta_{jk} m_{\alpha(i)} m_{\alpha(l)} + \delta_{jl} m_{\alpha(i)} m_{\alpha(k)} + \delta_{kl} m_{\alpha(i)} m_{\alpha(j)}) + \frac{1}{2}(\bar{\epsilon}^2 - \frac{(\bar{\epsilon}^2 + 2)\bar{\mu}^4}{8}) (\delta_{ij} \delta_{kl} + \delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) \right), \quad (29)$$

then

$$E[\hat{\mathcal{M}}] = (1 - \frac{\bar{\epsilon}^2}{2})(1 - 3(\bar{\epsilon}^2 - \frac{\bar{\mu}^4}{8}))\bar{\mathcal{M}}. \quad (30)$$

Since multiplication by a constant does not affect the eigenmatrix, the unit eigenmatrix \mathbf{Q} of $\hat{\mathcal{M}}$ for the smallest eigenvalue is statistically unbiased.

However, if we are to do this computation, we again face the same problems that we encountered for intersection estimation. Namely, the optimal weights (27)

contain the conic \mathbf{Q} that we want to fit, and evaluation of $\hat{\mathcal{M}}$ requires evaluation of the mean square magnitude $\bar{\epsilon}^2$ and the fourth-order moment $\bar{\mu}^4$, which are difficult to predict a priori for real images in real environments.

4.3 Renormalization

These difficulties can be overcome by exactly the same technique that we applied to intersection estimation. Firstly, the optimal weights W_α are determined by iterations. Ignoring the fourth-order quantities $\bar{\epsilon}^4$ and $\bar{\mu}^4$ and putting $c = \bar{\epsilon}^2$, we can express the tensor $\hat{\mathcal{M}} = (\hat{M}_{ijkl})$ in the form

$$\hat{M}_{ijkl} = (1 - \frac{\bar{\epsilon}^2}{2}) \sum_{\alpha=1}^N W_\alpha (m_{\alpha(i)} m_{\alpha(j)} m_{\alpha(k)} m_{\alpha(l)} - \frac{c/2}{1 - c/2} (\delta_{ij} m_{\alpha(k)} m_{\alpha(l)} + \delta_{ik} m_{\alpha(j)} m_{\alpha(l)} + \delta_{il} m_{\alpha(k)} m_{\alpha(j)} + \delta_{jk} m_{\alpha(i)} m_{\alpha(l)} + \delta_{jl} m_{\alpha(i)} m_{\alpha(k)} + \delta_{kl} m_{\alpha(i)} m_{\alpha(j)})) \quad (31)$$

Ideally, the constant c should be determined so that $E[\hat{\mathcal{M}}] = \bar{\mathcal{M}}$, but this is impossible unless image noise characteristics are known. On the other hand, if $E[\hat{\mathcal{M}}] = E[\bar{\mathcal{M}}]$, eq. (30) suggests that

$$E[(\bar{\mathbf{Q}}, \hat{\mathcal{M}}\bar{\mathbf{Q}})] = (\bar{\mathbf{Q}}, E[\hat{\mathcal{M}}]\bar{\mathbf{Q}}) \propto (\bar{\mathbf{Q}}, \bar{\mathcal{M}}\bar{\mathbf{Q}}) = 0. \quad (32)$$

Note that from eq. (24) $J(\mathbf{Q}) = (\mathbf{Q}, \mathcal{M}\mathbf{Q})$ takes its absolute minimum 0 for the exact solution $\bar{\mathbf{Q}}$. Hence, it is reasonable to choose c so that $(\mathbf{Q}, \hat{\mathcal{M}}\mathbf{Q}) = 0$ in each iteration step.

Since the constant multiplier $1 - \bar{\epsilon}^2/2$ does not affect the eigenmatrix of $\hat{\mathcal{M}}$, it can be dropped from the above definition. Renaming $(c/2)/(1 - c/2)$ as $c/2$, we obtain from eq. (31)

$$(\mathbf{Q}, \hat{\mathcal{M}}\mathbf{Q}) = (\mathbf{Q}, \mathcal{M}\mathbf{Q}) - c(\text{tr}\mathbf{Q}\text{tr}(\mathbf{M}\mathbf{Q}) + 2\text{tr}(\mathbf{M}\mathbf{Q}^2)), \quad (33)$$

where \mathbf{M} is the moment matrix of $\{\mathbf{m}_\alpha\}$ defined by

$$\mathbf{M} = \sum_{\alpha=1}^N W_\alpha \mathbf{m}_\alpha \mathbf{m}_\alpha^\top. \quad (34)$$

If $(\mathbf{Q}, \hat{\mathcal{M}}\mathbf{Q}) \neq 0$ for the current estimates c and \mathbf{Q} , then

$$(\mathbf{Q}, \hat{\mathcal{M}}\mathbf{Q}) - c'(\text{tr}\mathbf{Q}\text{tr}(\mathbf{M}\mathbf{Q}) + 2\text{tr}(\mathbf{M}\mathbf{Q}^2)) = 0 \quad (35)$$

for

$$c' = \frac{(\mathbf{Q}, \hat{\mathcal{M}}\mathbf{Q})}{\text{tr}\mathbf{Q}\text{tr}(\mathbf{M}\mathbf{Q}) + 2\text{tr}(\mathbf{M}\mathbf{Q}^2)}. \quad (36)$$

Note that $(\mathbf{Q}, \hat{\mathcal{M}}\mathbf{Q})$ equals the smallest eigenvalue of $\hat{\mathcal{M}}$.

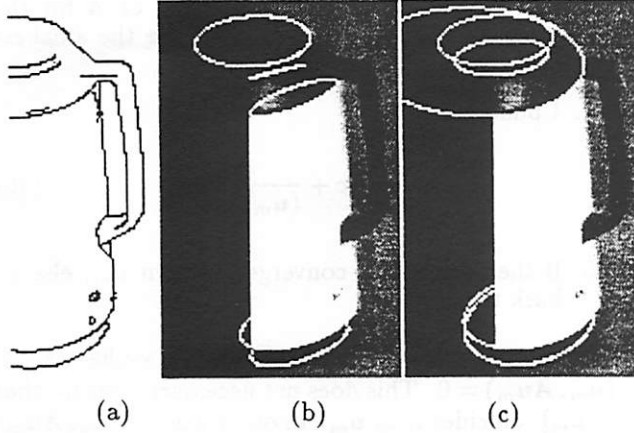


Figure 6: (a) A half edge image. (b) Least-squares fitting. (c) Renormalization.

From this observation, we obtain the following procedure of renormalization:

renormalization($\{m_\alpha\}$)

1. Let $c = 0$ and $Q = -I/\sqrt{3}$.
2. Compute

$$W_\alpha = \frac{1/\|Qm_\alpha\|^2}{\sum_{\beta=1}^N 1/\|Qm_\beta\|^2}. \quad (37)$$

3. Compute matrix $M = (M_{ij})$ and tensor $\hat{M} = (\hat{M}_{ijkl})$ by

$$M_{ij} = \sum_{\alpha=1}^N W_\alpha m_{\alpha(i)} m_{\alpha(j)}, \quad (38)$$

$$\begin{aligned} \hat{M}_{ijkl} = \sum_{\alpha=1}^N W_\alpha & (m_{\alpha(i)} m_{\alpha(j)} m_{\alpha(k)} m_{\alpha(l)} \\ & - \frac{c}{2} (\delta_{ij} m_{\alpha(k)} m_{\alpha(l)} + \delta_{ik} m_{\alpha(j)} m_{\alpha(l)} \\ & + \delta_{il} m_{\alpha(k)} m_{\alpha(j)} + \delta_{jk} m_{\alpha(i)} m_{\alpha(l)} \\ & + \delta_{jl} m_{\alpha(i)} m_{\alpha(k)} + \delta_{kl} m_{\alpha(i)} m_{\alpha(j)})). \end{aligned} \quad (39)$$

4. Let Q be the unit eigenmatrix of \hat{M} for the smallest eigenvalue, and let λ_m be the smallest eigenvalue.
5. Update c by

$$c \leftarrow c + \frac{\lambda_m}{\text{tr} Q \text{tr}(MQ) + \text{tr}(MQ^2)}. \quad (40)$$

6. Return Q if the update has converged; else go back to Step 2.

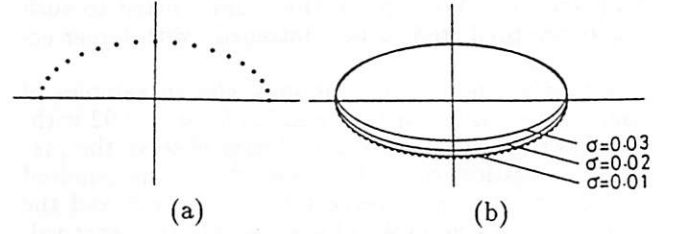


Figure 7: (a) Points on the upper half of a conic. (b) Theoretically predicted bias.

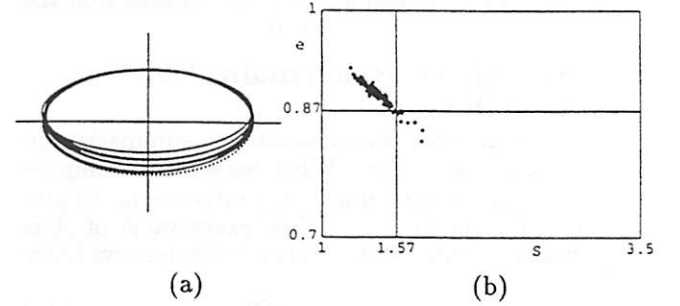


Figure 8: Conic fitting without renormalization. (a) Ten samples of fitted conics. (b) One hundred samples of the area S and the eccentricity e .

Example. Fig. 6(a) is an edge image obtained from a 300×200 -pixel real image. In order to magnify the differences, the original image is cut half. Fig. 6(b) shows the fits obtained by least squares with optimal weights, and Fig. 6(c) shows the fits obtained by renormalization. Since edges are short, the fits are not accurate, but renormalization still produces better results. In all cases, renormalization converges after three or four iterations.

Fig. 7(a) shows nineteen points on the upper half of $x^2 + 4y^2 = 1$. The x and y coordinates of each point are displaced by independent Gaussian noise with standard deviation σ ($f = 10$). Fig. 7(b) shows the expectation of the conic theoretically predicted [11, 14] for $\sigma = 0.01, 0.02, 0.03$. The exact conic is indicated by

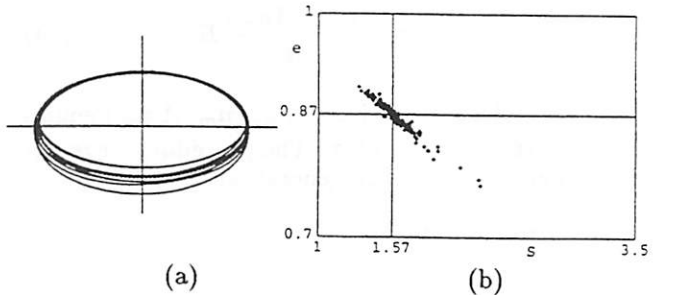


Figure 9: Conic fitting with renormalization. (a) Ten samples of fitted conics. (b) One hundred samples of the area S and the eccentricity e .

a broken line. We can see that conics fitted to such points are predicted to be "flattened" with larger eccentricities.

Fig. 8(a) shows ten randomly chosen samples of conics fitted with constant weights for $\sigma = 0.02$ without renormalization. We can clearly observe the predicted statistical bias. Fig. 8(b) shows one hundred samples plotted with respect to the area S and the eccentricity e (the broken lines indicate the exact values).

Fig. 9 show the corresponding result obtained by applying renormalization. The computation converges after three or four iterations. It is clearly seen that the statistical bias has been reduced.

5 Generalized Renormalization

5.1 Definition

The procedure for renormalization is summarized in abstract terms as follows. What we want to compute is the unit eigenvector \tilde{u}_m of a positive semi-definite matrix \hat{A} for eigenvalue 0. The exact value of \hat{A} is unknown, but from a statistical error analysis we know that

$$\bar{A} = E[A - cB], \quad (41)$$

where A and B are symmetric matrices we can compute from image data, while c is an unknown constant characterizing the behavior of image noise. Matrices A and B are random variables, since they are algebraically computed from image data, while c has a definite value determined by the statistical model of image noise. Hence, if we put

$$\hat{A} = A - cB, \quad (42)$$

and if we can choose c such that $E[\hat{A}] = \bar{A}$, the unit eigenvector u_m of \hat{A} for the smallest eigenvalue is an unbiased estimate of \tilde{u}_m . However, we cannot do this unless we know noise characteristics. On the other hand, if $E[\hat{A}] = \bar{A}$, then

$$E[(\tilde{u}_m, \hat{A}\tilde{u}_m)] = (\tilde{u}_m, E[\hat{A}]\tilde{u}_m) = (\tilde{u}_m, \bar{A}\tilde{u}_m) = 0. \quad (43)$$

So, we attempt to compute a unit vector u_m such that $(u_m, \hat{A}u_m) = 0$. If $(u_m, Au_m) \neq 0$ for the current estimate of c , we define

$$\hat{A}' = \hat{A} - \frac{(u_m, \hat{A}u_m)}{(u_m, Bu_m)}B. \quad (44)$$

Then, $(u_m, \hat{A}'u_m) = 0$. Note that $(u_m, \hat{A}'u_m)$ equals the smallest eigenvalue of \hat{A} . The procedure for renormalization is described in general terms as follows:

renormalization(A, B)

1. Let $c = 0$.
2. Compute

$$\hat{A} = A - cB. \quad (45)$$

3. Compute the unit eigenvector u_m of \hat{A} for the smallest eigenvalue, and let λ_m be the smallest eigenvalue.

4. Update c by

$$c \leftarrow c + \frac{\lambda_m}{(u_m, Bu_m)}. \quad (46)$$

5. If the update has converged, return u_m ; else go back to Step 2.

If u_m and c are the converged values, we have $\lambda_m = (u_m, \hat{A}u_m) = 0$. This does not necessarily ensure that $E[u_m]$ coincides with \tilde{u}_m . In other words, $(u_m, \bar{A}u_m)$ is not necessarily 0, because the current image data are not necessarily "typical" (i.e., a good representative of the statistical ensemble). However, we can expect that u_m is a good approximation with a high probability.

5.2 Speed of convergence

If u_m is the unit eigenvector of the current \hat{A} for the smallest eigenvalue λ_m , matrix \hat{A} is updated at the next step to

$$\hat{A}' = \hat{A} - \frac{\lambda_m}{(u_m, Bu_m)}B. \quad (47)$$

According to the perturbation theorem, the smallest eigenvalue λ'_m of \hat{A}' is

$$\lambda'_m = \lambda_m - (u_m, \left(\frac{\lambda_m}{(u_m, Bu_m)}B \right) u_m) + O\left(\left(\frac{\lambda_m}{(u_m, Bu_m)}B \right)^2 \right) = O(\lambda_m^2), \quad (48)$$

where $O(\dots)^2$ denotes terms of order 2 or higher in \dots . This means that λ_m converges to 0 *quadratically* just like Newton iterations, and the number of significant digits approximately doubles at each iteration. Hence, the convergence is very rapid, and three or four iterations are sufficient for most cases.

Let λ_1 and λ_2 be, respectively, the largest and the second largest eigenvalues of \hat{A} , and u_1 and u_2 the corresponding unit eigenvectors. According to the perturbation theorem, the unit eigenvector u'_m of \hat{A}' at the next step for the smallest eigenvalue λ'_m is

$$\begin{aligned} u'_m &= u_m + \frac{\lambda_m}{(u, Bu)} \sum_{i=1,2} \frac{(u_i, Bu_m)}{\lambda_m - \lambda_i} u_i + O(\lambda_m^2) \\ &= u_m + O(\lambda_m). \end{aligned} \quad (49)$$

Since λ_m converges to 0 quadratically, the convergence of u_m is also quadratic. The optimal weights W_α were computed by using the current eigenvector u_m , and since the convergence of λ_m is quadratic, the convergence of the W_α is also quadratic.

Although convergence of renormalization is very rapid, it should be emphasized that the converged values are not necessarily the *exact* values, because the noise is random and unpredictable. The purpose of renormalization is to remove statistical bias (not completely, though), so the resulting estimates still have errors (approximately of mean 0). It can be generally proved [11, 12] that if each datum has an independent error of root-mean-square magnitude ν and if the number of data is N , the optimal unbiased estimate has an error of root-mean-square magnitude $O(\nu/\sqrt{N})$, which is the lowest bound that can be achieved.

5.3 Kalman filtering

Kalman filtering is also an effective method of statistical estimation. It is a "linearized" update rule for modifying the current estimate by a linear operation ("orthogonal projection" to be precise) each time a new datum is added, so that the updated estimate is optimal over the data so far read. Hence, as many iterations as the number of data are necessary. In contrast, renormalization is a "nonlinear" update rule (computing eigenvectors and eigenvalues) over the entire data. Hence, the number of iterations is independent of the number of data; usually, three or four iterations are sufficient because the convergence is quadratic. Kalman filtering can be modified to fit many types of problems, but it does not make sense to force Kalman filtering to those problems which admit renormalization.

6 Concluding Remarks

In this paper, we have studied the problem of estimating vanishing points and focuses of expansion and conic fitting. Employing a statistical model of image noise, we showed that the least-squares solution is "statistically biased" in the presence of noise, and presented a scheme called "renormalization", which iteratively removes the statistical bias by automatically adjusting to the image noise. Its effectiveness was demonstrated by real image experiments and random number simulations. We also proved that the speed of convergence of renormalization is quadratic and compared renormalization with Kalman filtering.

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