Int. J. Engng Sci. Vol. 22, No. 5, pp. 531–546, 1984
Printed in Great Britain.

0020-7225/84 $3.00 + .00

STEREOLOGICAL DETERMINATION OF STRUCTURAL ANISOTROPY

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(Communicated by SHUN-ICHI AMARI)

Abstract—A general mathematical formulation is given to the problem of determining the structural anisotropy by means of the stereological principle. Three cases are considered—distributed curves in the plane, distributed curves in the space and distributed surfaces in the space. The number of intersections with a probe line or plane is viewed as a transformation, which is termed the "Buffon transform", between two distribution densities, and a form of its inverse transform is given. Then, the change of anisotropy due to the deformation of the material is formulated, and the strain is shown to be determined from the data of the intersection counting. All equations are written in the form of Cartesian tensor equations invariant to coordinate translations and rotations. A typical example is also given.

1. INTRODUCTION

SO CALLED "integral geometry" or "stereology" has provided a useful tool to measure geometrical characteristics of distributed curves and surfaces by counting the number of intersections on a probe line or plane [1–10]. For example, if the probe line or plane is placed randomly, the number of intersections is proportionally related to the specific length and the specific area of the distributed curves and surfaces respectively. The study of this kind is old and stems from "Buffon's needle problem" [11]. If intersections are counted for each different orientation of the probe line or plane, then we can also determine the "structural anisotropy", i.e. the distribution of the curves or surfaces. This problem was analyzed by Hilliard [12], and explicit formulae were presented to determine the "distribution density".

On the other hand, if the problem under consideration is a "physical" one—like anisotropic structures of metals and composite materials, the description of anisotropy must be expressed in "frame indifferent forms" [13, 14], i.e. forms invariant to coordinate translations and rotations. This is because the coordinate system to which parameters of orientation are referred is completely arbitrary. Thus, any equation describing material properties must have the same form irrespective of the choice of the coordinate system, which implies that it must be a Cartesian tensor equation. Since the structural anisotropy is completely specified by the distribution density, the distribution density itself must be expressed in terms of Cartesian tensors. This problem was studied by Kanatani [15]. He described distribution densities of directional data in terms of what he called the "fabric tensors", applying orthogonality properties of functions on a sphere or a circle, which is also related to the representation theory of the rotation group and the spherical harmonics expansion or the Fourier series expansion. He also gave explicit formulae to determine these tensors from the moments of given data and a procedure of the statistical test of fitness, applying the asymptotic theory of statistical testing.

In this paper, we give a Cartesian tensor formulation to Hilliard's method by applying Kanatani's analysis. This makes it possible to generalize different types of problems into a single mathematical framework called the "Buffon transform". Then, we give a form of its inverse transform in the form of Cartesian tensor equations, which makes it possible to determine the tensors characterizing the distribution density, or the "fabric tensors", directly from the data of intersection counting. A typical example is given to illustrate the technique. Finally, we consider the change of structural anisotropy due to the deformation of the material. This problem was already studied by Philofsky and Flinn [16]. Here, however, the change of the distribution density is characterized by transformations of fabric tensors and is written as tensor equations. This enables one to know the strain tensor of the material directly from the data of intersection counting.

2. BUFFON TRANSFORM OF DISTRIBUTION DENSITY

Suppose curves are distributed in the plane. Let the curves be dissected into infinitesimal line elements of length dl and each line element choose one of the two unit normal vectors $\boldsymbol{n}$ or $-\boldsymbol{n}$ independently with probability 1/2. The "distribution density" $f(\boldsymbol{n})$ is defined in such a way that $f(\boldsymbol{n})\,d\boldsymbol{n}$ is the "total length" of those line elements whose normal vector is lying between $\boldsymbol{n}$ and $\boldsymbol{n}+d\boldsymbol{n}$ per unit area. Here, $d\boldsymbol{n}$ is a symbolic notation for the differential angle. If we use a particular polar coordinate θ, then $d\boldsymbol{n}=d\theta$. However, as was discussed in Section 1, we adopt "coordinate independent" notations throughout this paper. Obviously $f(\boldsymbol{n})$ is "symmetric", i.e., $f(\boldsymbol{n})=f(-\boldsymbol{n})$, and $\int f(\boldsymbol{n})\,d\boldsymbol{n}$ is the total length of the curves per unit area.

Let a line of direction $\boldsymbol{m}$ (a unit vector) be placed randomly in this plane, and consider the number of intersections with the curves. Put the expected number of intersections per unit length to be $N(\boldsymbol{m})$.

Theorem 2.1

$$N(\boldsymbol{m})=\int |\boldsymbol{m}\cdot\boldsymbol{n}|f(\boldsymbol{n})\,d\boldsymbol{n}. \tag{2.1}$$

Proof. The probability that a line element with its normal between $\boldsymbol{n}$ and $d\boldsymbol{n}$ intersects the line equals the probability that the center of the line element falls within the distance $|\boldsymbol{m}\cdot\boldsymbol{n}|dl/2$ from the line (Fig. 1). Since there are $f(\boldsymbol{n})\,d\boldsymbol{n}/dl$ such line elements per unit area, that probability is $|\boldsymbol{m}\cdot\boldsymbol{n}|f(\boldsymbol{n})\,d\boldsymbol{n}$ per unit length of the line, which proves the theorem.

Here, we are not interested in "spatial correlations" of the distribution but considering only the "expected number of intersections". Hence, the above discussion is rigorous. If we repeat this "line dropping" independently a large number of times and take the average over all the trials, then the l.h.s. of eqn (2.1) can be interpreted in the sense of this data average due to the "law of large number". Obviously, $N(\boldsymbol{m})$ is also symmetric, i.e. $N(\boldsymbol{m})=N(-\boldsymbol{m})$. We term a transformation of a distribution density of the form of eqn (2.1) the "Bufforn transform" after his pioneering work of this problem more than two centuries ago[11].

Now, turn to the case of distributed surfaces in the space. Again, we treat the mutually opposite normal vectors randomly with probability 1/2. Let $f(\boldsymbol{n})$ be the distribution density of surface elements with unit normal $\boldsymbol{n}$, i.e., $f(\boldsymbol{n})\,d\boldsymbol{n}$ is the "total area" of those surface elements whose normal vector is lying in the differential solid angle $d\boldsymbol{n}$ per unit volume. (If particular spherical coordinates θ, ϕ are used, then $d\boldsymbol{n}=\sin\theta\,d\theta\,d\phi$). Obviously, $f(\boldsymbol{n})=f(-\boldsymbol{n})$ and $\int f(\mathrm{n})\,d\boldsymbol{n}$ is the total area of the surfaces per unit volume. Place a line of direction $\boldsymbol{m}$ (a unit vector) randomly in the space, and let $N(\boldsymbol{m})$ be the expected number of intersections with the surfaces per unit length.

Theorem 2.2

$$N(\boldsymbol{m})=\int |\boldsymbol{m}\cdot\boldsymbol{n}|f(\boldsymbol{n})d\boldsymbol{n}. \tag{2.2}$$

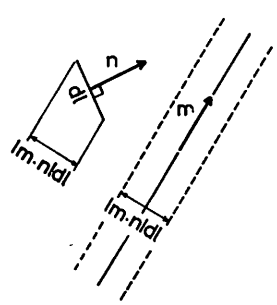

Fig. 1. A line element intersects the line when its center falls within the distance $|\boldsymbol{m}\cdot\boldsymbol{n}|dl/2$ from the line.

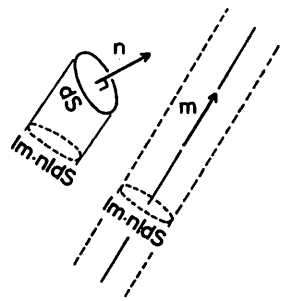

Fig. 2. A surface element intersects the line when its center falls inside a cylinder of base area $|\boldsymbol{m}\cdot\boldsymbol{n}|\mathrm{d}S$.

Proof. The probability that a surface element of area $\mathrm{d}S$ with its normal lying in the differential solid angle $\mathrm{d}\boldsymbol{n}$ intersects the line equals the probability that the center of the surface element falls inside the "cylinder" of base area $|\boldsymbol{m}\cdot\boldsymbol{n}|\mathrm{d}S$ along the line (Fig. 2). Since there are $f(\boldsymbol{n})\mathrm{d}\boldsymbol{n}/\mathrm{d}S$ such surface elements per unit volume, that probability is $|\boldsymbol{m}\cdot\boldsymbol{n}|f(\boldsymbol{n})\mathrm{d}\boldsymbol{n}$ per unit length of the line. Hence, $N(\boldsymbol{m})$ is given by the three dimensional Buffon transform.

Finally, consider distributed curves in the space. Choosing the direction of each unit "tangent" vector randomly as before, let $f(\boldsymbol{n})$ be its distribution density of line elements with unit tangent vector $\boldsymbol{n}$, i.e. $f(\boldsymbol{n})\mathrm{d}\boldsymbol{n}$ is the "total length" of those line elements whose tangent vector is lying in the differential solid angle $\mathrm{d}\boldsymbol{n}$ per unit volume. Hence, $f(\boldsymbol{n})=f(-\boldsymbol{n})$, and $\int f(\boldsymbol{n})\mathrm{d}\boldsymbol{n}$ is the total length of the curves per unit volume. Place a plane whose unit normal is $\boldsymbol{m}$ randomly in the space, and let $N(\boldsymbol{m})$ be the expected number of intersections with the curves per unit area.

Theorem 2.3

$$N(\boldsymbol{m})=\int|\boldsymbol{m}\cdot\boldsymbol{n}|f(\boldsymbol{n})\mathrm{d}\boldsymbol{n}. \tag{2.3}$$

Proof. The probability that a line element of length $\mathrm{d}l$ with its tangent lying in the differential solid angle $\mathrm{d}\boldsymbol{n}$ intersects the plane equals the probability that the center of the line element falls within the distance $|\boldsymbol{m}\cdot\boldsymbol{n}|\mathrm{d}l/2$ from the plane (Fig. 3). Since there are $f(\boldsymbol{n})\mathrm{d}\boldsymbol{n}/\mathrm{d}l$ such line elements per unit volume, that probability is $|\boldsymbol{m}\cdot\boldsymbol{n}|f(\boldsymbol{n})\mathrm{d}\boldsymbol{n}$ per unit area of the plane. Hence $N(\boldsymbol{m})$ is again given by the Buffon transform of eqn (2.3).

3. INVERSE BUFFON TRANSFORM—3-DIMENSIONAL CASE

It was shown in the previous section that the distribution density of structural anisotropy is related to the number of intersections on the probe line or plane by the Buffon transform. Therefore, if its inverse transform is known, the distribution density can be estimated by placing the probe line or plane in various different orientations and

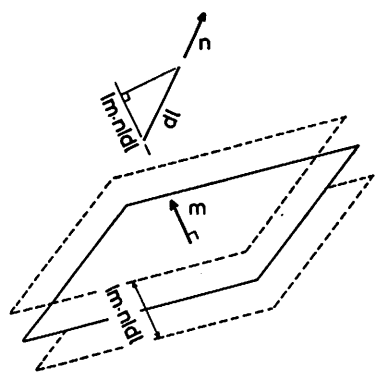

Fig. 3. A line element intersects the plane when its center falls within the distance $|\boldsymbol{m}\cdot\boldsymbol{n}|\mathrm{d}l/2$ from the plane.

counting the number of intersections on it. The simplest and the most practical way to obtain the inversion formula is to express the distribution density in a parametric form. Since we are seeking a frame indifferent form, the parametric form must be a Cartesian tensor equation. This problem was fully studied by Kanatani[15]. He expanded the distribution density into a polynomial in $\boldsymbol{n}$ in such a way that each term is orthogonal to the rest. In other words, it is a Cartesian tensor expression of the spherical harmonics expansion in the 3-dimensional case and the Fourier series expansion in the 2-dimensional case. He called the coefficient tensors "fabric tensors". We first consider the 3-dimensional case. Hereafter, we adopt the summation convention over repeated tensor indices.

A symmetric distribution density $f(\boldsymbol{n})$ is expressed in the form

$$f(\boldsymbol{n}) = \frac{C}{4\pi}[1 + D_{ij}n_in_j + D_{ijkl}n_in_jn_kn_l + \ldots], \tag{3.1}$$

where C and the "fabric tensor" $D_{i_1\ldots i_n}$ are given respectively by

$$C = \int f(\boldsymbol{n})\mathrm{d}\boldsymbol{n}, \tag{3.2}$$

$$D_{i_1\ldots i_n} = \frac{2n+1}{2^n}\binom{2n}{n}N_{\{i_1\ldots i_n\}}, \tag{3.3}$$

and $N_{i_1\ldots i_n}$ is the "moment tensor" defined by

$$N_{i_1\ldots i_n} = \frac{1}{C}\int n_{i_1}\ldots n_{i_n}f(\boldsymbol{n})\mathrm{d}\boldsymbol{n}. \tag{3.4}$$

Here, $\{\ \}$ designates the "deviator part", i.e., the combination of the components such that any contraction makes it vanish. In general, if $A_{i_1\ldots i_n}$ is a symmetric tensor, its deviator part is given by

$$\begin{aligned} A_{\{i_1\ldots i_n\}} = {} & c_0^nA_{i_1\ldots i_n} + c_2^n\delta_{(i_1i_2}A_{i_3\ldots i_n)ij} + c_4^n\delta_{(i_1i_2}\delta_{i_3i_4}A_{i_5\ldots i_n)kkij} \\ & + \ldots + c_n^n\delta_{(i_1i_2}\delta_{i_3i_4}\cdots\delta_{i_{n-1}i_n)}A_{j_1j_1j_2j_2\ldots j_{n/2}j_{n/2}}, \end{aligned} \tag{3.5}$$

where δ_{ij} is the Kronecker delta and $()$ designates the symmetrization of tensor indices. Here, c_m^n's are defined by

$$c_m^n = (-1)^{m/2}\binom{n}{m}\binom{n-1}{m/2}\Big/\binom{2n-1}{m}. \tag{3.6}$$

This formulation is very practical, because we can calculate all quantities from the moment tensor of observed data. We discuss actual practical procedures later.

Since the fabric tensor $D_{i_1\ldots i_n}$ is a deviator tensor, eqn (3.1) can be also written as

$$f(\boldsymbol{n}) = \frac{C}{4\pi}[1 + D_{ij}n_{\{i}n_{j\}} + D_{ijk}n_{\{i}n_jn_{k\}} + \ldots]. \tag{3.7}$$

(It can be easily checked that $n_{\{i_1}\ldots n_{i_n\}}$ is in fact a spherical harmonic of degree n [15].) Since the Buffon transform is a "linear" transformation, the Buffon transform of eqn (3.7) is obtained once the Buffon transform of $n_{\{i_1}\ldots n_{i_n\}}$ is known. The following theorem plays a fundamental role.

Theorem 3.1

$$\int|\boldsymbol{m}\cdot\boldsymbol{n}|n_{\{i_1}\ldots n_{i_n\}}\mathrm{d}\boldsymbol{n} = \frac{2\pi}{a_n}m_{\{i_1}\ldots m_{i_n\}}, \tag{3.8}$$

where a_n is a constant.

Proof. Note that the Buffon transform "commutes" with rotations. Namely, consider the function space $L^2(S^2)$ on a unit sphere S^2 and put $\boldsymbol{B}f(\boldsymbol{n}) = \int |\boldsymbol{n} \cdot \boldsymbol{n}'| f(\boldsymbol{n}')\, d\boldsymbol{n}'$ and $\boldsymbol{R}^* f(\boldsymbol{n}) = f(\boldsymbol{R}^{-1}\boldsymbol{n})$ for $f(\boldsymbol{n}) \in L^2(S^2)$, where $\boldsymbol{R}$ is a 3-dimensional rotation. Then, $\boldsymbol{B}\boldsymbol{R}^* f(\boldsymbol{n}) = \int |\boldsymbol{n} \cdot \boldsymbol{n}'| f\boldsymbol{R}^{-1}\boldsymbol{n}')\, d\boldsymbol{n}' = \int |\boldsymbol{n} \cdot \boldsymbol{R}\boldsymbol{n}'| f(\boldsymbol{n}')\, d\boldsymbol{n}' = \int |\boldsymbol{R}^T\boldsymbol{n} \cdot \boldsymbol{n}'| f(\boldsymbol{n}')\, d\boldsymbol{n}' = \int |\boldsymbol{R}^{-1}\boldsymbol{n} \cdot \boldsymbol{n}'| f(\boldsymbol{n}')\, d\boldsymbol{n}' = \boldsymbol{R}^* \boldsymbol{B} f(\boldsymbol{n})$ because $d\boldsymbol{n}$ is invariant to rotations. Since $\boldsymbol{B}^*\boldsymbol{R} = \boldsymbol{R}^*\boldsymbol{B}$ for any $\boldsymbol{R}$, or $\boldsymbol{B}$ is an "invariant operator", the subspace W_n of $L^2(S^2)$ spanned by $n_{\{i_1} \ldots n_{i_n\}}$'s, which is a representation space of an irreducible representation of the rotation group, is also an eigenspace of $\boldsymbol{B}$. (This is easily shown by the orthogonality of irreducible representations and Schur's lemma). Hence follows eqn (3.8), and $2\pi/a_n$ is the eigenvalue. Alternatively, we can show it by noting that an invariant operator is expressed as a power series of the "Laplace–Beltrami operator" $\Lambda = (1/\sin\theta)(\partial/\partial\theta)\sin\theta\,\partial/\partial\theta + (1/\sin^2\theta)\,\partial^2/\partial\phi^2$. The subspace W_n is an eigenspace of Λ and its elements are spherical harmonics of degree n. Hence, W_n is also an eigenspace of $\boldsymbol{B}$.

In order to determine the eigenvalue $2\pi/a_n$, we can take an arbitrary coordinate system, and it is sufficient to calculate a particular component or any linear combination of those components of the same n, i.e. we can take any function which belongs to W_n. Let us take a particular Cartesian coordinate system such that the z-axis coincides with the direction of $\boldsymbol{m}$ and use the associated spherical coordinates θ, ϕ to describe the direction of $\boldsymbol{n}$. As an eigenfunction, we can take the Legendre function $P_n(\cos\theta)$ as a representative, since it is a vector of W_n. Then, we have

$$\int_0^{2\pi}\int_0^{\pi} |\cos\theta| P_n(\cos\theta)\sin\theta\, d\theta\, d\phi = \frac{2\pi}{a_n} P_n(1) \left(= \frac{2\pi}{a_n}\right). \tag{3.9}$$

The l.h.s. becomes $4\pi \int_0^1 z P_n(z)\, dz$. Hence, using the Rodrigues formula and integration by parts, we obtain

Theorem 3.2

$$a_n = (-1)^{n/2-1} 2^{n-1}(n-1)(n+2) \Big/ \binom{n}{n/2}. \tag{3.10}$$

For example, $a_0 = 1$, $a_2 = 4$, $a_4 = -24$, $a_6 = 64, \ldots$. Hence, we obtain the following two theorems.

Theorem 3.3.

The Buffon transform of eqn (3.1) is

$$N(\boldsymbol{m}) = \frac{2\pi C}{4\pi}\left[1 + \frac{1}{4} D_{ij} m_i m_j - \frac{1}{24} D_{ijkl} m_i m_j m_k m_l + \ldots\right]. \tag{3.11}$$

Theorem 3.4.

Conversely, if $N(\boldsymbol{m})$ is given in the form

$$N(\boldsymbol{m}) = \frac{C}{4\pi}[1 + D_{ij} m_i m_j + D_{ijkl} m_i m_j m_k m_l + \ldots], \tag{3.12}$$

then its inverse Buffon transform is given by

$$f(\boldsymbol{n}) = \frac{C/2\pi}{4\pi}[1 + 4 D_{ij} n_i n_j - 24 D_{ijkl} n_i n_j n_k n_l + \ldots]. \tag{3.13}$$

Expression $N(\boldsymbol{m})$ in the form of eqn (3.12) is easy. The simplest way is to choose the orientation $\boldsymbol{m}$ of the probe line or plane randomly, i.e. according to the uniform distribution over a unit sphere. Suppose that $\boldsymbol{m}^{(1)}, \ldots, \boldsymbol{m}^{(N)}$ are the selected probe orientations and that $N(\boldsymbol{m}^{(1)}), \ldots, N(\boldsymbol{m}^{(n)})$ are the corresponding observed data. Then, we can estimate C by

$$C = \langle 1 \rangle, \tag{3.14}$$

where $\langle . \rangle = 4\pi \sum_{\alpha=1}^{N} (.) N(\boldsymbol{m}^{(\alpha)})/N$ (weighted average). The moment tensor $N_{i_1 \ldots i_n}$ is estimated by

$$N_{i_1 \ldots i_n} = \frac{1}{C} \langle n_{i_1} \ldots n_{i_n} \rangle. \tag{3.15}$$

(Equations (3.14) and (3.15) are nothing but evaluation of eqns (3.2) and (3.4) by the "Monte–Carlo method"). Then, the fabric tensor $D_{i_1 \ldots i_n}$ is given by eqn (3.3). Once the fabric tensor is known, the distribution density $f(\boldsymbol{n})$ is given by eqn (3.13). It is sufficient to take up to the second term in most applications. If necessary, we can perform a statistical test of fitness by Kanatani's procedure [15]. Moreover, in most of related physical problems, what is important is not the form of $f(\boldsymbol{n})$ itself but rather its fabric tensors themselves, notably that of the second rank D_{ij}.

If the probe orientations are chosen not uniformly over a unit sphere but according to some probability density $p(\boldsymbol{m})$, the same procedure is available except that each $N(\boldsymbol{m}^{(\alpha)})$ is divided by $4\pi p(\boldsymbol{m}^{(\alpha)})$. If, on the other hand, the probing is performed for $\boldsymbol{m}^{(1)}, \ldots, \boldsymbol{m}^{(N)}$ without any reference to probability, which is the most likely situation for most experiments, we can estimate the density $p(\boldsymbol{m})$ by the same technique[15]. Namely, put

$$N_{i_1 \ldots i_n} = \frac{1}{N} \sum_{\alpha=1}^{N} m_{i_1}^{(\alpha)} \ldots m_{i_n}^{(\alpha)}, \tag{3.16}$$

and calculate $D_{i_1 \ldots i_n}$ by eqn (3.3). Then $p(\boldsymbol{m})$ is given by

$$p(\boldsymbol{m}) = \frac{1}{4\pi} [1 + D_{ij} m_i m_j + D_{ijkl} m_i m_j m_k m_l + \ldots]. \tag{3.17}$$

This turns out the "least square error approximation" [15]. Again, it is sufficient to take up to the second term in most cases, resorting to the above mentioned statistical test of fitness, if necessary. (If we keep adding higher terms indefinitely, $p(\boldsymbol{m})$ approaches its "empirical density", i.e. a discontinuous function with as many singularities as the data).

4. INVERSE BUFFON TRANSFORM—2-DIMENSIONAL CASE

In the 2-dimensional case, a symmetric distribution $f(\boldsymbol{n})$ is expressed in the form

$$f(\boldsymbol{n}) = \frac{C}{2\pi} [1 + D_{ij} n_i n_j + D_{ijkl} n_i n_j n_k n_l + \ldots], \tag{4.1}$$

where C and the fabric tensor $D_{i_1 \ldots i_n}$ are given, respectively, by

$$C = \int f(\boldsymbol{n}) \mathrm{d}\boldsymbol{n}, \tag{4.2}$$

$$D_{i_1 \ldots i_n} = 2^n N_{\{i_1 \ldots i_n\}}, \tag{4.3}$$

and $N_{i_1 \ldots i_n}$ is the moment tensor defined by eqn (3.4). The deviator part of a symmetric tensor $A_{i_1 \ldots i_n}$ also has the form of eqn (3.5) in the 2-dimensional case, but c_m^n's in this case are given by

$$c_m^n = \frac{(-1)^{m/2}}{2^m} \frac{n}{n - m/2} \binom{n - m/2}{m/2}. \tag{4.4}$$

The tensor $n_{\{i_1} \ldots n_{i_n\}}$ turns out a circular harmonic function of degree n, i.e. a sinusoidal function of argument $n\theta$. Hence, eqn (4.1) is nothing but a Cartesian tensor equation of the Fourier series expansion of $f(\boldsymbol{n})$ [15]. As in the 3-dimensional case, we have

Theorem 4.1

$$\int |\boldsymbol{m} \cdot \boldsymbol{n}| n_{\{i_1} \dots n_{i_n\}} \mathrm{d}\boldsymbol{n} = \frac{4}{a_n} m_{\{i_1} \dots m_{i_n\}}, \tag{4.5}$$

where $4/a_n$ is the eigenvalue.

Proof. Let W_n be the linear space generated by $n_{\{i_1} \dots n_{i_n\}}$'s. It is 2-dimensional and spanned by $e^{in\theta}$ and $e^{-in\theta}$ or by $\cos n\theta$ and $\sin n\theta$. The proof of Theorem 3.1 does not apply as it is, because Schur's lemma holds only for "algebraically closed fields". In other words, the group representation theoretical properties of Theorem 3.1 is valid only in the complex domain. This is no problem in the 3-dimensional case, but in the 2-dimensional case any irreducible representation of the rotaton group is of degree 1 in the complex domain and the basis is $e^{in\theta}$ or $e^{-in\theta}$ ($n = 0, 1, 2, \dots$), which must be an eigenvector of the Buffon transform $\boldsymbol{B}$. However, as long as we resort to the group theory, there is no guarantee that $e^{in\theta}$ and $e^{-in\theta}$ share the same eigenvalue, though the eigenvalues must be a complex conjugate pair because $\boldsymbol{B}$ is a real operator. In other words, the eigenvalue may be complex and the "phase shift" may arise. However, as will be shown by a direct calculation, this is not the case. The eigenvalue is real and there is no phase shift. Hence W_n is an eigenspace of $\boldsymbol{B}$. (Recall the theory of control. A "time invariant" or "stationary" linear system, i.e. one whose operation "commutes" with the operation of time shift, has $e^{i\omega t}$ as an eigenfunction, because it spans the 1-dimensional space of an irreducible representation of the group of time shift, i.e. 1-dimensional translations. Its eigenvalue is complex in general and called the "frequency response", its modulus being the "gain" and its argument being the "phase shift").

In order to determine the eigenvalue, let us take a Cartesian coordinate system such that the x-axis coincides with the direction of $\boldsymbol{m}$ and use the associated polar coordinate θ to describe the direction of $\boldsymbol{n}$. As an integrand, consider $e^{in\theta}$. Then, we have

$$\int_0^{2\pi} |\cos\theta| (\cos n\theta + i \sin n\theta) \mathrm{d}\theta = \frac{4}{a_n}. \tag{4.6}$$

Note, however, that the term of $\sin n\theta$ vanishes because it is an odd function in θ while $|\cos\theta|$ is an even one. Hence, the eigenvalue is real, and $e^{-in\theta} = \cos n\theta - i \sin n\theta$ also has the same eigenvalue. Thus, we have

Theorem 4.2

$$a_n = (-1)^{n/2+1}(n^2 - 1). \tag{4.7}$$

For example, $a_0 = 1$, $a_2 = 3$, $a_4 = -15$, $a_6 = 35, \dots$. Hence, we obtain

Theorem 4.3

The Buffon transform of eqn (4.1) is given by

$$N(\boldsymbol{m}) = \frac{4C}{2\pi}\left[1 + \frac{1}{3} D_{ij} m_i m_j - \frac{1}{15} D_{ijkl} m_i m_j m_k m_l + \dots\right]. \tag{4.8}$$

Theorem 4.4

Conversely, if $N(\boldsymbol{m})$ is given in the form

$$N(\boldsymbol{m}) = \frac{C}{2\pi}[1 + D_{ij} m_i m_j + D_{ijkl} m_i m_j m_k m_l + \dots], \tag{4.9}$$

then its inverse Buffon transform is given by

$$f(\boldsymbol{n}) = \frac{C/4}{2\pi}[1 + 3D_{ij} n_i n_j - 15 D_{ijkl} n_i n_j n_k n_l + \dots]. \tag{4.10}$$

The actual procedure is the same as in the 3-dimensional case. If the orientation $\boldsymbol{m}$ of the probe is chosen according to the uniform distribution over a unit circle, then C and the moment tensor $N_{i_1 \ldots i_n}$ are estimated by eqns (3.14) and (3.15), respectively, where the weighted average is in this case $\langle . \rangle = 2\pi \sum_{\alpha=1}^{N} (.) N(\boldsymbol{m}^{(\alpha)})/N$. Then, the fabric tensor $D_{i_1 \ldots i_n}$ is determined by eqn (4.3) and $f(\boldsymbol{n})$ is given by eqn (4.1). Here again, it is sufficient to take up to the second term in most applications, or, more precisely, our interest is often not in the form of $f(\boldsymbol{n})$ but D_{ij} itself. If $\boldsymbol{m}$ is chosen according to some probability density $p(\boldsymbol{m})$, the procedure is the same if $N(\boldsymbol{m}^{(\alpha)})$ is divided by $2\pi p(\boldsymbol{m}^{(\alpha)})$ in the weighted average. If the probe directions are chosen without reference to any probability, the density $p(\boldsymbol{m})$ is estimated as before. First, compute $N_{i_1 \ldots i_n}$ by eqn (3.16) and then $D_{i_1 \ldots i_n}$ by eqn (4.3). Then, $p(\boldsymbol{m})$ is given by

$$p(\boldsymbol{m}) = \frac{1}{2\pi} [1 + D_{ij} m_i m_j + D_{ijk} m_i m_j m_k m_l + \ldots]. \tag{4.11}$$

This is again the least square error approximation. In most applications, it is sufficient to take up to the second term.

Hilliard also proposed the use of a curved test array with a prescribed orientation distribution for manual intersection counting [12]. In this case, however, we have to use a particular coordinate system, because the coordinate system to describe the orientation distribution of the test array and the coordinate system for the material are different and independent. Hence, the Cartesian tensor expression cannot be used to describe his procedure. For details, see [12]. On the other hand, the "parallel line scanning" is the most practical procedure for manual calculation, and it is most easily implemented by a computer image processing system on the graphic display screen. Today, we cannot think of the full use of stereological principles without computers. In fact, it can be said that the stereological principles will find their true appreciation only when computers are used for implementation.

As an example, consider the pattern of Fig. 4, which is obtained after distorting an initially isotropic random patter drawn by generating random numbers. This can be viewed as a simulation of grain boundaries on a cross section of a polycrystalline metal which has undergone a deformation. The data of intersection counting are shown in Fig. 5. (We used equally spaced parallel lines, among which 27 intersected the circumference circle). The solid curve in Fig. 5 shows the curve $N(\boldsymbol{m}) = C[1 + D_{ij} m_i m_j]/2\pi$ fitted as stated above. (The data and the curve are normalized so that $C = 1$). The fabric tensor D_{ij} becomes

$$D_{ij} = \begin{bmatrix} 0.1786 & -0.0109 \\ -0.0109 & -0.1786 \end{bmatrix}. \tag{4.12}$$

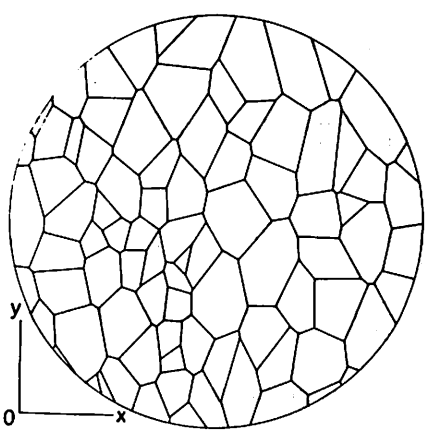

Fig. 4. An artificial pattern simulating grain boundaries on a cross section of a deformed polycrystalline metal.

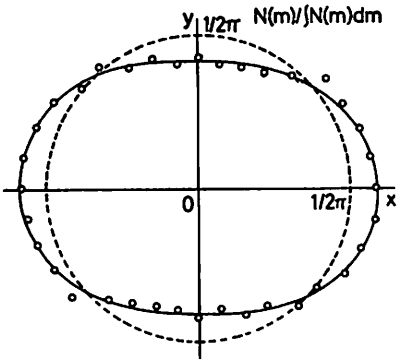

Fig. 5. Normalized data of intersection counting on Fig. 4. and its fitted curve.

Thus, the distribution density of the line segments is given by $f(\boldsymbol{n}) = (C/4)[1 + 3D_{ij}n_in_j]/2\pi$. Figure 6 shows the normalized density $f(\boldsymbol{n})/\int(\boldsymbol{n})d\boldsymbol{n}$. (Note that $\boldsymbol{n}$ is the unit "normal", not the "tangent").

5. FABRIC CHANGE DUE TO DEFORMATION—CURVES IN THE SPACE

We now consider the change of structural anisotropy induced by the deformation of the material. First, consider the case of distributed curves in the space. Suppose the material undergoes a deformation described by a linear transformation

$$x_i' = A_{ij}x_j, \tag{5.1}$$

and let $f(\boldsymbol{n})$ be the distribution density of the curves before the deformation. A unit vector n_i is deformed to $A_{ij}n_j$, and it is no longer a unit vector. Its length becomes

$$L(\boldsymbol{n}) = \sqrt{A_{ki}A_{kj}n_in_j}. \tag{5.2}$$

Hence, the unit vector $\boldsymbol{n}'$ describing its direction is given by

$$n_i' = \frac{1}{L(\boldsymbol{n})}A_{ij}n_j. \tag{5.3}$$

Equation (5.3) is viewed as a map on a unit sphere on which the distribution density is defined. Hence, we must calculate the "Jacobian" $d\boldsymbol{n}'/d\boldsymbol{n}$ of the map in order to know the transformed density. This is done as follows. Consider a transformation

$$x_i' = \sqrt{\frac{x_nx_n}{A_{mk}A_{ml}x_kx_l}}A_{ij}x_j. \tag{5.4}$$

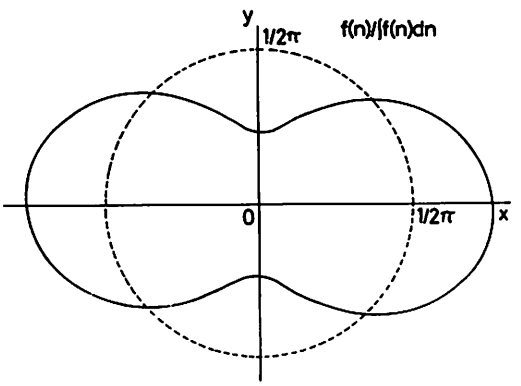

Fig. 6. The distribution density of line segments of Fig. 4 obtained by the inverse Buffon transform of Fig. 5. (The vector $\boldsymbol{n}$ is the unit "normal", not the "tangent").

The restriction of this transformation on a unit sphere coincides with eqn (5.3), and the distance from the origin is conserved by this transformation. Hence, in order to know the area $d\boldsymbol{n}'$ after transformation (5.3) of an infinitesimal region $d\boldsymbol{n}$ on the sphere, we have only to consider the volume of a "cylinder" of base $d\boldsymbol{n}$ and with infinitesimal height after transformation (5.4). In other words, it is sufficient to calculate the Jacobian of transformation (5.4) at $\boldsymbol{x} = \boldsymbol{n}$. After some manipulations, we obtain

$$\frac{d\boldsymbol{n}'}{d\boldsymbol{n}}(\boldsymbol{n}) = \det(\partial x_i'/\partial x_j)|_{x=n} = \det A/L(\boldsymbol{n})^3. \tag{5.5}$$

Hence, the new distribution density is given by

$$f(\boldsymbol{n}') = \frac{L(\boldsymbol{n})}{\det A} f(\boldsymbol{n}) \Big/ \frac{d\boldsymbol{n}'}{d\boldsymbol{n}}(\boldsymbol{n}). \tag{5.6}$$

In order to obtain a more practical form, let us consider only small deformations and put

$$A_{ij} = \delta_{ij} + F_{ij}, \tag{5.7}$$

and assume that F_{ij} is small. The tensor F_{ij} is called the "distortion tensor" and further resolved into the "symmetric part" and the "skew symmetric part", namely

$$F_{ij} = e_{ij} + r_{ij}, \tag{5.8}$$

$$e_{ij} = F_{(ij)}\left(= \frac{1}{2}(F_{ij} + F_{ji})\right), \tag{5.9}$$

$$r_{ij} = F_{[ij]}\left(= \frac{1}{2}(F_{ij} - F_{ji})\right), \tag{5.10}$$

where () and [] designate the symmetrization and the alternation of the tensor indices respectively. The tensor e_{ij} is the "strain tensor"and r_{ij} is the "rotation tensor". The strain tensor e_{ij} is further resolved into the "scalar part" and the "deviator part", namely

$$e_{ij} = \tilde{e}_{ij} + \frac{1}{3} e_{kk}\delta_{ij}, \tag{5.11}$$

$$\tilde{e}_{ij} = e_{\{ij\}}\left(= e_{ij} - \frac{1}{3} e_{kk}\delta_{ij}\right). \tag{5.12}$$

The tensor $\tilde{e}_{ij}$ is known to be the "shear strain tensor" and e_{kk} the "volumetric strain". As is well known, these resolution is invariant to coordinate transformations.

Using these notations, and neglecting higher order terms, we obtain

$$L(\boldsymbol{n}) = 1 + \frac{1}{3} e_{kk} + \tilde{e}_{ij} n_i n_j + 0(\boldsymbol{F}^2), \tag{5.13}$$

$$n_i' = n_i + \tilde{e}_{ij} n_j + r_{ij} n_j - \tilde{e}_{kl} n_k n_l n_i + 0(\boldsymbol{F}^2), \tag{5.14}$$

$$\frac{d\boldsymbol{n}'}{d\boldsymbol{n}} = 1 - 3\tilde{e}_{ij} n_i n_j + 0(\boldsymbol{F}^2), \tag{5.15}$$

and $\det A = 1 + e_{kk} + 0(\boldsymbol{F}^2)$, where $0(\boldsymbol{F}^2)$ designates terms of F_{ij} whose order is equal to and higher than 2. The following theorem describes the practically most important case.

Theorem 5.1

If the initial distribution density is given by

$$f(\boldsymbol{n}) = \frac{c}{4\pi}[1 + D_{ij}n_in_j], \tag{5.16}$$

then, the distribution density after the deformation becomes

$$f(\boldsymbol{n}') = \frac{c'}{4\pi}[1 + D'_{ij}n'_in'_j + D'_{ijkl}n'_in'_jn'_kn'_l] + 0(F^2), \tag{5.17}$$

$$c' = c\left(1 - \frac{2}{3}e_{kk} + \frac{2}{15}\tilde{e}_{ij}D_{ij}\right), \tag{5.18}$$

$$D'_{ij} = D_{ij} + 8\tilde{e}_{ij} + \frac{10}{7}\tilde{e}_{(i|k}D_{k|j)} - \frac{2}{15}\tilde{e}_{kl}D_{kl}D_{ij} - \frac{10}{21}\delta_{ij}\tilde{e}_{kl}D_{kl} - 2r_{(i|k}D_{k|j)}, \tag{5.19}$$

$$D'_{ijkl} = 6\tilde{e}_{(ij}D_{kl)} - \frac{24}{7}\delta_{(ij}\tilde{e}_{k|m}D_{m|l)} + \frac{12}{35}\delta_{(ij}\delta_{kl)}\tilde{e}_{mn}D_{mn}, \tag{5.20}$$

where $|\ |$ designates the exclusion from the symmetrization operation.

These are first approximations, or, more precisely, we have obtained "infinitesimal generators". In other words, we have calculated $\mathrm{d}/\mathrm{d}t|_{t=0}$ for "one parameter subgroups" $\exp(e_{kk}It/3)$, $\exp(\tilde{e}t)$ and $\exp(rt)$ respectively of the "deformation group" $\mathrm{GL}^+(3, \boldsymbol{R})$, which is a Lie subgroup of $\mathrm{GL}(3, \boldsymbol{R})$ consisting of those linear transformations whose determinant is positive. For example, $\mathrm{d}c/\mathrm{d}t|_{t=0} = -2e_{kk}/3$ for $A(t) = \exp(e_{kk}It/3)$. Equations (5.18)–(5.20) must be interpreted in this sense. (Here, $e_{kk}\delta_{ij}/3$, $\tilde{e}_{ij}$ and r_{ij} span the "Lie algebra" of $GL^+(3, \boldsymbol{R})$. This Lie algebra completely generates $\mathrm{GL}^+(3, \boldsymbol{R})$ because it is a "connected" Lie group).

In particular, if the initial structure is isotropic, i.e. $f(\boldsymbol{n}) = c/4\pi$, the distribution density after the deformation becomes

$$f(\boldsymbol{n}') = \frac{c(1 - 2e_{kk}/3)}{4\pi}[1 + 4\tilde{e}_{ij}n'_in'_j] + 0(F^2). \tag{5.21}$$

Hence, if we use the stereological procedure discussed in the previous sections, we can easily determine the strain tensor as follows.

Theorem 5.2

If the initial distribution density is $f(\boldsymbol{n}) = c/4\pi$ and the data of intersection counting is given by $N(\boldsymbol{m}) = C[1 + D_{ij}m_im_j + \ldots]/4\pi$, the strain as a first approximation is given by

$$\tilde{e}_{ij} = D_{ij}, \quad e_{kk} = \frac{3}{2}\left(1 - \frac{C}{2\pi c}\right). \tag{5.22}$$

If the initial structure is not isotropic but its distribution density $f(\boldsymbol{n})$ is known in the form of eqn (5.16), then we can deduce the strain from the two distribution densities before and after the deformation. However, as is seen from eqns (5.18)–(5.20), e_{kk}, $\tilde{e}_{ij}$ and r_{ij} are coupled with each other, and hence we need a prior knowledge about the coupling. If, for example, the material does not rotate, then eqn (5.19) provides a complete set of equations to solve $\tilde{e}_{ij}$ in terms of D_{ij} and D'_{ij}. Then, eqn (5.18) determines e_{kk} in terms of known c, c' and D_{ij}. If, on the other hand, the material rotates without strains, then eqn (5.19) determines the rotation r_{ij}.

6. FABRIC CHANGE DUE TO DEFORMATION—SURFACES IN THE SPACE

We now consider the case of distributed surfaces in the space. Let $f(\boldsymbol{n})$ be the distribution density of the surfaces and consider the deformation (5.1) again. Let n_{ij} be the

skew symmetric tensor describing a unit surface ($n_{ij}n_{ij}/2 = 1$). Then, the tensor after deformation (5.1) becomes

$$n'_{ij} = A_{ik}A_{jl}n_{kl}, \tag{6.1}$$

which does not describe a unit surface any more. The unit normal $\boldsymbol{n}$ to the initial surface is given by

$$n_i = \frac{1}{2}\epsilon_{ijk}n_{jk}, \tag{6.2}$$

where ϵ_{ijk} is the Eddington epsilon, i.e. the signature of permutation $(123)\rightarrow(ijk)$. The unit normal to the deformed surface is, therefore, given by

$$n'_i = \frac{1}{2S(\boldsymbol{n})}\epsilon_{ikl}\epsilon_{jmn}A_{km}A_{ln}n_j. \tag{6.3}$$

$$S(\boldsymbol{n}) = \sqrt{\epsilon_{ikl}\epsilon_{jmn}A_{pk}A_{ql}A_{pm}A_{qn}n_in_j/2}. \tag{6.4}$$

Equation (6.3) defines a map on a unit sphere. The transformed density is calculated in the same way as in the previous section. If the deformation is small, we obtain after neglecting higher order terms

$$S(\boldsymbol{n}) = 1 + \frac{1}{3}e_{kk} - \tilde{e}_{ij}n_in_j + 0(\boldsymbol{F}^2), \tag{6.5}$$

$$n'_i = n_i - \tilde{e}_{ij}n_j + r_{ij}n_j + \tilde{e}_{jk}n_jn_kn_i + 0(\boldsymbol{F}^2), \tag{6.6}$$

$$\frac{\mathrm{d}\boldsymbol{n}'}{\mathrm{d}\boldsymbol{n}} = 1 + 3\tilde{e}_{ij}n_in_j + 0(\boldsymbol{F}^2), \tag{6.7}$$

where we have used the notations defined in the previous section.

Theorem 6.1

If the initial distribution density has the form

$$f(\boldsymbol{n}) = \frac{c}{4\pi}[1 + D_{ij}n_in_j], \tag{6.8}$$

the distribution density after the deformation becomes

$$f(\boldsymbol{n}') = \frac{c'}{4\pi}[1 + D'_{ij}n'_in'_j + D'_{ijkl}n'_in'_jn'_kn'_l] + 0(\boldsymbol{F}^2), \tag{6.9}$$

$$c' = c\left(1 - \frac{1}{3}e_{kk} - \frac{2}{15}\tilde{e}_{ij}D_{ij}\right), \tag{6.10}$$

$$D'_{ij} = D_{ij} - 4\tilde{e}_{ij} - \frac{10}{7}\tilde{e}_{(i|k}D_{k|j)} + \frac{2}{15}\tilde{e}_{kl}D_{kl}D_{ij} + \frac{10}{21}\delta_{ij}\tilde{e}_{kl}D_{kl} - 2r_{(i|k}D_{k|j)}, \tag{6.11}$$

$$D'_{ijkl} = -6\tilde{e}_{(ij}D_{kl)} + \frac{24}{7}\delta_{(ij}\tilde{e}_{k|m}D_{m|l)} - \frac{12}{35}\delta_{(ij}\delta_{kl)}\tilde{e}_{mn}D_{mn}. \tag{6.12}$$

In particular, if the initial structure is isotropic, i.e. $f(\boldsymbol{n}) = c/4\pi$, the distribution density after the deformation becomes

$$f(\boldsymbol{n}') = \frac{c(1 - e_{kk}/3)}{4\pi}[1 - 4\tilde{e}_{ij}n_in_j] + 0(\boldsymbol{F}^2). \tag{6.13}$$

Hence, the strain tensor is directly connected with the data of intersection counting by

Theorem 6.2

If the initial distribution density if $f(\boldsymbol{n}) = c/4\pi$ and the data of intersection counting is given by $N(\boldsymbol{m}) = C[1 + D_{ij}m_im_j + \ldots]/4\pi$, the strain as a first approximation is given by

$$\tilde{e}_{ij} = -D_{ij}, \quad e_{kk} = 3\left(1 - \frac{C}{2\pi c}\right). \tag{6.14}$$

If the initial distribution is not isotropic but its distribution density $f(\boldsymbol{n})$ is known, the procedure discussed in the previous section can be also applied.

7. FABRIC CHANGE DUE TO DEFORMATION—CURVES IN THE PLANE

Finally, consider the case of distributed curves in the plane, and let $f(\boldsymbol{n})$ be the distribution density of the curves. We again consider the linear transformation of eqn (5.1) in the 2-dimensional space. Let $\boldsymbol{t}$ be the unit tangent to a line element. It is deformed into

$$t_i' = A_{ij}t_j, \tag{7.1}$$

which is no longer a unit vector. The unit normal $\boldsymbol{n}$ to the initial line element is given by

$$n_i = \epsilon_{ij}t_j, \tag{7.2}$$

where ϵ_{ij} is the signature of permutation (12)→(ij). The unit normal to the deformed line element is, therefore, given by

$$n_i' = \frac{1}{L(\boldsymbol{n})}\epsilon_{ik}\epsilon_{jl}A_{kl}n_j, \tag{7.3}$$

$$L(\boldsymbol{n}) = \sqrt{\epsilon_{ik}\epsilon_{jl}A_{mk}A_{ml}n_in_j}. \tag{7.4}$$

Equation (7.3) defines a map on a unit circle. The transformed density is calculated in the same way as before. If the deformation is small, we obtain after neglecting higher order terms

$$L(\boldsymbol{n}) = 1 + \frac{1}{2}e_{kk} - \tilde{e}_{ij}n_in_j + 0(F^2), \tag{7.5}$$

$$n_i' = n_i - \tilde{e}_{ij}n_j + r_{ij}n_j + \tilde{e}_{jk}n_jn_kn_i + 0(F^2), \tag{7.6}$$

$$\frac{\mathrm{d}\boldsymbol{n}'}{\mathrm{d}\boldsymbol{n}} = 1 + 2\tilde{e}_{ij}n_in_j + 0(F^2), \tag{7.7}$$

where, as before, we have put $A_{ij} = \delta_{ij} + F_{ij}$, $e_{ij} = F_{(ij)}$, $r_{ij} = F_{[ij]}$ and $\tilde{e}_{ij} = e_{ij} - e_{kk}\delta_{ij}/2$.

Theorem 7.1

If the initial distribution density has the form

$$f(\boldsymbol{n}) = \frac{c}{2\pi}[1 + D_{ij}n_in_j], \tag{7.8}$$

the distribution density after the deformation becomes

$$f(\boldsymbol{n}') = \frac{c'}{2\pi}[1 + D_{ij}'n_i'n_j' + D_{ijkl}'n_j'n_j'n_k'n_l'] + 0(F^2), \tag{7.9}$$

$$c' = c\left(1 - \frac{1}{2}e_{kk} - \frac{1}{4}\tilde{e}_{ij}D_{ij}\right), \tag{7.10}$$

$$D'_{ij} = D_{ij} - 3\tilde{e}_{ij} + \frac{1}{4}\tilde{e}_{kl}D_{kl}D_{ij} - 2D_{ik}r_{kj}, \tag{7.11}$$

$$D'_{ijkl} = -5\tilde{e}_{(ij}D_{kl)} + \frac{10}{3}\delta_{(ij}\tilde{e}_{k|m}D_{m|l)} - \frac{5}{12}\delta_{(ij}\delta_{kl)}\tilde{e}_{mn}D_{mn}. \tag{7.12}$$

In particular, if the initial structure is isotropic, i.e. $f(\boldsymbol{n}) = c/2\pi$, the distribution density after the deformation becomes

$$f(\boldsymbol{n}') = \frac{C(1 - e_{kk}/2)}{2\pi}[1 - 3\tilde{e}_{ij}n_in_j] + 0(F^2). \tag{7.13}$$

Hence, the strain tensor is directly connected with the data of intersection by

Thorem 7.2

If the initial distribution density is $f(\boldsymbol{n}) = c/2\pi$ and the data of intersection counting is given by $N(\boldsymbol{m}) = C[1 + D_{ij}m_im_j + \ldots]/4\pi$, the strain as a first approximation is given by

$$\tilde{e}_{ij} = -D_{ij}, \quad e_{kk} = 2\left(1 - \frac{C}{4c}\right). \tag{7.14}$$

If the initial distribution is not isotropic, the previous discussion also applies.

Let us consider the example of Fig. 4 again. Suppose this pattern is obtained by a small deformation of an initially isotropic pattern. Then, in view of eqn (4.12) and Theorem 7.2, the shear strain tensor is given by

$$\tilde{e}_{ij} = \begin{bmatrix} -0.1786 & 0.0109 \\ 0.0109 & 0.1786 \end{bmatrix}. \tag{7.15}$$

The eigenvalues are ± 0.1789 and the directions of corresponding eigenvectors are $\theta = 91.8°$, $1.8°$ from the x-axis, respectively. Hence, if we take now x'- and y'-axes, rotating the x- and y-axes by 1.8°, we obtain

$$\tilde{e}_{i'j'} = \begin{bmatrix} -0.1789 & 0 \\ 0 & 0.1789 \end{bmatrix}. \tag{7.16}$$

Therefore, if we extend the pattern 1.4 times along the x'-axis, we will obtain a pattern which is approximately isotropic.

8. CONCLUDING REMARKS

We first established the relationship between the distribution density of distributed lines or surfaces and the data of intersection counting in a single form which we called the Buffon transform and then gave a form of its inverse transform. Next, we gave a form of the change of distribution densities due to the deformation of the material and related the data of intersection counting to the strain of the material.

As a matter of fact, the results formulated here are essentially not new, as was mentioned in Introduction, and there are many studies which show calculations that are essentially the same as ours. However, in most cases, these calculations are done in reference to a specific spherical or polar coordinate system, and distribution densities are expressed in the Laplace spherical harmonics expansion or the Fourier series expansion. In contrast, we presented in this paper all the formulations in coordinate independent Cartesian tensor equation forms. The advantage is three-fold.

First, we can obtain general expressions in a very compact form, which reveals equivalences among seemingly different problems. Thus, we obtained the concept of the Buffon transform irrespective of the geometry of distributed figures and the spatial dimensionality, which leads to a common analytical technique to handle it despite the

difference in physical interpretation. Second, describing a problem in the coordinate free form often saves otherwise tedious calculations involving specific coordinates and functions (sin's, cos's and associated Legendre functions, for example). Thus, we obtained a form of the inverse Buffon transform by noting that the operation of the Buffon transform on functions on a unit sphere commutes with that of rotations and by applying the group theoretical properties. Third, since any physical laws must be expressed in frame indifferent forms, the stereological principles expressed in frame indifferent forms can be connected with physical properties readily. Thus, we showed that the data of intersection counting expressed in the Cartesian tensor form is directly related to the strain tensor of the material.

The theory generalized in this paper has a wide variety of applications. Consider the example of Fig. 4, for instance. It is not a mere illustrating example. We are simulating a test of grain boundaries of polycrystalline metals. In any experiment of mechanical properties of metals, the experimenter must carefully choose a specimen to avoid the effect of factors outside the present consideration, and the past history of deformation is one of them. Therefore, he must first test if a given specimen is free from past deformations, and if not he must annhilate the past history, say, by annealation. Then, he must check whether the past history has completely disappeared or not. These tests are most easily done by observing the metal surface by a microscope and by applying the procedure we showed in this paper. The same technique is also applied to computer image processing—for example, measuring the orientation of a surface from a projected image and recovering the true surface pattern by cancelling the effect of projective distortion[17].

Since computers are coming into use in physical experiments more and more these days, it seems that stereological principles like those given here will be indispensable and play a major role in physical observations and data processings.

Acknowledgement—Part of this research was supported by Saneyoshi Scholarship Foundation.

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(*Received* 7 *May* 1983)

APPENDIX—SECOND FORM OF THE BUFFON TRANSFORM

In the text, we considered three cases, i.e. distributed curves in the plane vs. a probe line, distributed surfaces in the space vs. a probe line and distributed curves vs. a probe plane. There is, however, another possibility left. i.e. distributed surfaces vs. a probe plane. What is measured, in this case, is the "length" of the cross-section curves per unit area of the probe plane. Let $f(\boldsymbol{n})$ be the distribution density of the surfaces and $\boldsymbol{m}$ be the unit normal to the probe plane. Then, the expected length $L(\boldsymbol{m})$ of the cross-section curves per unit area of the probe plane is given by

$$L(\boldsymbol{m}) = \int \|\boldsymbol{m} \times \boldsymbol{n}\| f(\boldsymbol{n}) \mathrm{d}\boldsymbol{n}. \tag{A1}$$

This is obtained by the following observation. Consider those surface elements with area dS whose normals lie in the differential solid angle $d\boldsymbol{n}$ around $\boldsymbol{n}$. There are $f(\boldsymbol{n})d\boldsymbol{n}/dS$ such surface elements per unit volume. Obviously, the expected length of the cross section of the probe plane with unit normal $\boldsymbol{m}$ and such a surface element is the same as that of the cross-section of the same probe plane and an image of that surface element orthographically projected onto a plane spanned by $\boldsymbol{m}$ and $\boldsymbol{m} \times \boldsymbol{n}$. The area of the projected surface element is $\|\boldsymbol{m} \times \boldsymbol{n}\| dS$, and it coincides with the average cross-section length per unit area of the probe plane if it vertically cuts the unit area containing the image at random.

Turn to the case of distributed curves in the plane. In the text, the distribution density $f(\boldsymbol{n})$ was defined with respect to the "normal" $\boldsymbol{n}$ to a line element, and eqn (2.1) was obtained. If, on the other hand, the distribution density $f(\boldsymbol{n})$ is defined in terms of the "tangent" to a line element, the transformation of eqn (2.1) is rewritten as follows.

$$N(\boldsymbol{m}) = \int \|\boldsymbol{m} \times \boldsymbol{n}\| f(\boldsymbol{n}) d\boldsymbol{n}, \tag{A2}$$

where $\|\boldsymbol{m} \times \boldsymbol{n}\|$ denotes $|m_1 n_2 - m_2 n_1|$.

Eqn (A1) and (A2) give a second form of the Buffon transform. This transform maps $L^2(S^2)$ and $L^2(S^1)$ into themselves respectively and also commutes with rotations. Hence, it is an "invariant operator" and has $n_{(i_1} \ldots n_{i_n)}$ as an eigenvector. The corresponding eigenvalue is π^2/b_n for the 3-dimensional case and $4/b_n$ for the 2-dimensional case, where

$$b_n = \begin{cases} -2^{2n-1} \dfrac{n-1}{n} \Big/ \dbinom{n}{n/2}^2 & \text{(for 3 dim.)} \\ 1 - n^2 & \text{(for 2 dim.)} \end{cases}. \tag{A3}$$

(Caution: $b_0 = 1$.) Then, the same procedure as in the text applies, and this form of the Buffon transform also serves to determine the structural anisotropy and deformation of the material.