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PROCEDURES FOR STEREOLOGICAL ESTIMATION OF STRUCTURAL ANISOTROPY

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Abstract—Actual procedures are given for detecting structural anisotropy by the stereological method, i.e. by observing only cross-sections in the material, on the basis of the theoretical formulation of the previous article [1]. First, computation by the Monte Carlo method is described explicitly in terms of observed data, determining the distribution density completely. Next, alternative procedures are given by the use of restricted cross-sections on the assumption that the anisotropy is "weak," facilitating actual experiments a great deal. Observations are made only on planes parallel to the three coordinate planes or cylindrical surfaces around the three coordinate axes. "Fabric tensors" describing the anisotropy and the "equivalent strain" are given explicitly in terms of observed data.

1. INTRODUCTION

THIS IS a continuation of the previous study [1] of the stereological estimation of structural anisotropy. In [1], we considered three cases: lines and curves distributed on a two-dimensional plane, surfaces distributed in a three-dimensional material, and lines and curves distributed in a three-dimensional material. These are the most fundamental instances of structural anisotropy. The distributions of such internal structures are characterized by appropriately defined "distribution densities," but measurement of these densities according to their definitions are impractical, and in many cases impossible. As has been known, however, stereological procedures give indirect ways of measuring those densities. All we have to do is to count the number of intersections of the internal structure with randomly placed probe lines or planes (if the intersections are points) or to measure the length of intersections of the structure with randomly placed cutting planes (if the intersections are lines and curves).

These observed data are related to the distribution density by what is called the "Buffon transform" in [1], where it is formulated as a tensor equation and its inversion process is given on the basis of group theoretical considerations. The distribution density is characterized by what is called "fabric tensors" [2], and they are expressed explicitly in terms of observed data. Then, we can immediately know the "equivalent strain," i.e. the strain according to which the present anisotropy would be realized from an initially isotropic state. All these results are expressed as tensor equations invariant to coordinate transformations in [1]. Background of the problem is also reviewed there. As is pointed out in [1], the stereological procedure has significance not only in material science but also in many other fields. For example, the stereological technique is used to detect the orientation and motion of an object from its projected image on a plane of vision in the field of artificial intelligence and computer vision (Kanatani [3, 4]). In geology, the same technique is used to estimate the distribution of crack orientation in a rock mass, which determines the anisotropy of its mechanical properties, by observing cross-sections of the cracks appearing on a surface (Oda [5, 6] and Kananani [7]).

Since the discussion given in [1] is quite general and all equations are written in the form of tensor equations, we first describe actual procedures, using the Monte Carlo method implied there, in terms of particular polar or spherical coordinates to elucidate the results there. The Monte Carlo method is the most consistent way, in the sense that we can determine the distribution density to any degree of accuracy by letting the number N of observations be large enough. Meanwhile, if we are to carry out these procedures in an actual experiment, we immediately face a problem. For example, if we are to cut a material by planes of various orientations, we must prepare a large number of material samples, all of which are supposed to have the same anisotropy and choose, by generating random numbers, the orientations of the cross-sections to be observed. The experiment would be much easier if we only had to observe cross-sections, say, parallel to the xy-, yz-, and zx-planes with respect a fixed xyz-coordinate system instead.

Keeping this in mind, we next present a way of estimating structural anisotropy from restricted observations of this type alone, which will facilitate actual experiments a great deal. Of course, we cannot detect all sorts of anisotropy from restricted observations, because the information we obtain is insufficient. However, if we confine our consideration to "weak anisotropies" (whose precise definition is given later), restricted observations of this type are sufficient. In fact, there exist infinite ways of detecting the anisotropy from restricted observations if the anisotropy is weak, for the distribution density is characterized by a finite number of parameters. However, we must try to devise a method insensitive to possible random fluctuations of observed data. Here, we present a way in which two-dimensional analysis is done on three types of surfaces, and the necessary input data take the form of averages of a large number of observed values, so that possible errors may cancel out. The assumption of weak anisotropy does not restrict the scope of application, because most of the anisotropies we encounter in engineering problems are regarded as weak.

2. PROBING LINES AND CURVES IN TWO DIMENSIONS WITH LINES

Suppose lines and curves are distributed on a two-dimensional plane. They may be scattered as disjoint segments or form a connected mesh. Fix on the plane a Cartesian coordinate system. Place on the plane a line making an angle Θ from the x-axis randomly, and let $N(\Theta)$ be the expected number of intersections per unit length of the probe line with the lines and curves. Since $N(\Theta)$ is a periodic function in Θ with period 2π, it can be expanded into the following Fourier series.

$$N(\Theta) = \frac{C}{2\pi}[1 + \sum_{n=2}^{\infty}{}'(A_n \cos n\Theta + B_n \sin n\Theta)], \tag{2.1}$$

$$C = \int_0^{2\pi} N(\Theta)\mathrm{d}\Theta, \tag{2.2}$$

$$A_n = \frac{2}{C}\int_0^{2\pi} N(\Theta)\cos n\Theta \mathrm{d}\Theta, \qquad B_n = \frac{2}{C}\int_0^{2\pi} N(\Theta)\sin n\Theta \mathrm{d}\Theta. \tag{2.3}$$

Here, Σ' designates summation only with respect to even indices. Odd harmonics do not appear because $N(\Theta)$ is "symmetric" with respect to the origin, i.e. $N(\Theta) = N(\Theta + \pi)$. This Fourier series expansion can be expressed as a Cartesian tensor equation. If we put $\mathbf{m} = (\cos\Theta, \sin\Theta)^{\mathrm{T}}$, T standing for transpose, we obtain (see [2])

$$N(\mathbf{m}) = \frac{C}{2\pi}[1 + D_{ij}m_im_j + D_{ijkl}m_im_jm_km_l + \cdots]. \tag{2.4}$$

The Einstein summation convention over repeated indices is adopted throughout this article. In [2], the coefficient tensors are referred to as the "fabric tensors" of the data distribution, and $D_{i_1\ldots i_n}$ is expressed in terms of the nth Fourier series coefficients A_n and B_n. For example, D_{ij} is given by

$$(D_{ij}) = \begin{bmatrix} A_2 & B_2 \\ B_2 & -A_2 \end{bmatrix}. \tag{2.5}$$

The anisotropy of the structure is characterized by the distribution density $f(\theta)$ defined in such a way that $f(\theta)\mathrm{d}\theta$ is the total length of those line elements in unit area whose orientations are between θ and $\theta + \mathrm{d}\theta$. Here, θ and $\theta + \pi$ designate the same orientation of a line element, so that one of them is chosen randomly with a probability of $\frac{1}{2}$. Hence, $f(\theta)$ is "symmetric" with respect to the origin, i.e. $f(\theta) = f(\theta + \pi)$, and $c = \int_0^{2\pi} f(\theta)\mathrm{d}\theta$ is the total length of the lines and curves in unit area, which we call the "length density."

The distribution density $f(\theta)$ can also be expanded into a Fourier series, and it takes the form

$$f(\theta) = \frac{C/4}{2\pi}\left[1 - \sum_{n=2}^{\infty}{}' (n^2 - 1)(A_n \cos n\theta + B_n \sin n\theta)\right]. \qquad (2.6)$$

(See [1] for the proof.) In the Cartesian tensor form, this is written as

$$f(\mathbf{n}) = \frac{C/4}{2\pi}[1 - 3D_{ij}n_in_j - 15D_{ijkl}n_in_jn_kn_l + \cdots], \qquad (2.7)$$

where $\mathbf{n} = (\cos\theta, \sin\theta)^T$. (Note that θ and $\mathbf{n}$ designate the orientation of the "tangent" to a line element. In [1], they are used to designate the orientation of the "normal" to a line element. Hence, the signs of the coefficients in eqns (2.6) and (2.7) differ from those in [1].) Thus, in order to know the distribution density $f(\theta)$, we only have to know C and $D_{i_1 \ldots i_n}$'s. In particular, the length density is given by $c = C/4$.

The fabric tensor D_{ij} is related to the "equivalent strain" e_{ij} by

$$e_{ij} = -D_{ij}. \qquad (2.8)$$

Namely, the strain tensor of eqn (2.8) would yield the present anisotropy by deforming a material of initially isotropic internal structure if higher order terms of $f(\theta)$ are neglected. (See [1] for the proof.) Of course, superposition of any isotropic expansion or contraction does not change the fabric tensors, so that the equivalent strain is determined only up to the freedom of volume change.

In summary, the actual procedure goes as follows.

Procedure 1

Step 1. Draw equally spaced parallel lines of orientation $k\pi/N$, $k = 0, 1, \ldots, N$ on the plane, and let N_k be the number of intersections per unit length with the lines and curves.

Step 2. Compute the following A_n and B_n, $n = 2, 4, 6, \ldots$, corresponding to eqns (2.2) and (2.3) by

$$C = 2\pi \sum_{k=0}^{N-1} N_k/N, \qquad (2.9)$$

$$A_n = 2\sum_{k=0}^{N-1} N_k \cos(n\pi k/N) / \sum_{k=0}^{N-1} N_k, \qquad B_n = 2\sum_{k=0}^{N-1} N_k \sin(n\pi k/N) / \sum_{k=0}^{N-1} N_k. \qquad (2.10)$$

Step 3. The coefficient C is given by eqn (2.9), and the fabric tensor $D_{i_1 \ldots i_n}$ is given in terms of A_n and B_n as described in [2]. In particular, D_{ij} is given by eqn (2.5). The length density is given by $c = C/4$, and the distribution density $f(\theta)$ or $f(\mathbf{n})$ by eqn (2.6) or (2.7). The equivalent strain is given by eqn (2.8).

Thus, as long as the problem is two-dimensional, the procedure is straightforward and no random numbers are necessary. If we are interested in e_{ij}, and hence D_{ij}, the number N of probe orientations need not be large due to Shannon's sampling theorem, since we only have to obtain the second harmonics of $N(\Theta)$. As can be seen by comparing eqn (2.1) with eqn (2.6), measurement of $N(\Theta)$ amounts to application of a low-pass filter to the distribution density $f(\theta)$, attenuating high harmonics considerably.

Consider the synthetic pattern of Fig. 1, for example. The data of intersection counting are shown in Fig. 2, where $N = 18$, i.e. at 10° intervals, and the spacing of parallel lines is $\frac{1}{24}$ the diameter of the circumference. (In Fig. 2, the scale is normalized so that the average value becomes $\frac{1}{2}\pi$.) We obtain $A_2 = 0.026$ and $B_2 = 0.057$. Fig. 3 is the distribution density $f(\theta)$ estimated up to second Fourier harmonics. (Again, the scale is

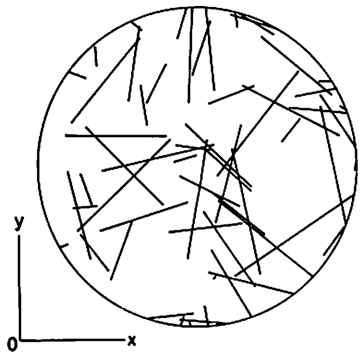

Fig. 1. Line segments scattered on a plane.

normalized so that the average value becomes $\frac{1}{2}\pi$.) It is difficult to obtain this result just by looking at Fig. 1. The equivalent strain tensor is given by

$$(e_{ij}) = \begin{bmatrix} -0.026 & -0.057 \\ -0.057 & 0.026 \end{bmatrix}. \tag{2.11}$$

The orientations of its principal axes are $\theta = 35.5°$, $125.5°$, and the corresponding principal strains are -0.188 and 0.188, respectively. Another example related to metallurgy is given in [1].

3. PROBING LINES AND CURVES IN THREE DIMENSIONS WITH PLANES

Consider a three-dimensional material in which lines and curves are distributed (as disjoint segments or a connected mesh or both). Cut the material randomly with a plane having a fixed normal. Let (Θ, Φ) be the spherical coordinates (associated with a fixed xyz-coordinate system) of the orientation of the normal, and let $N(\Theta, \Phi)$ be the expected number of intersections per unit area of the cutting plane with the lines and curves. Since it is a function of orientation, it can be expressed as a spherical harmonics expansion as follows:

$$N(\Theta, \Phi) = \frac{C}{4\pi}\Big[1 + \sum_{n=2}^{\infty}{}' \Big\{\tfrac{1}{2}A_{n0}P_n(\cos\Theta) + \sum_{m=1}^{n} P_n^m(\cos\Theta)[A_{nm}\cos m\Phi + B_{nm}\sin m\Phi]\Big\}\Big], \tag{3.1}$$

$$C = \int_0^{2\pi}\int_0^{\pi} N(\Theta, \Phi)\sin\Theta d\Theta d\Phi, \tag{3.2}$$

$$\begin{bmatrix} A_{nm} \\ B_{nm} \end{bmatrix} = \frac{2(2n+1)}{C}\frac{(n+m)!}{(n-m)!}\int_0^{2\pi}\int_0^{\pi} N(\Theta, \Phi)P_n^m(\cos\Theta)\begin{bmatrix} \cos m\Phi \\ \sin m\Phi \end{bmatrix}\sin\Theta d\Theta d\Phi. \tag{3.3}$$

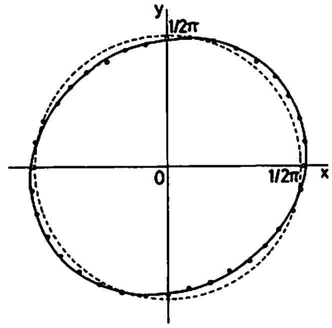

Fig. 2. Data of intersection counting normalized so that the average value becomes $\frac{1}{2}\pi$.

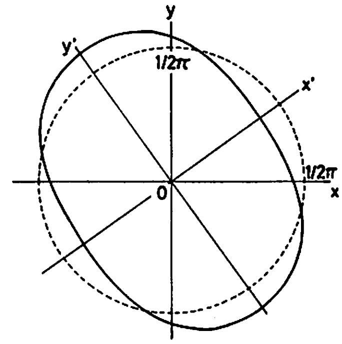

Fig. 3. Distribution density computed from the data of Fig. 2 up to second harmonics and normalized so that the average becomes $\frac{1}{2}\pi$.

Here, $P_n(z)$ is the nth Legendre polynomial and $P_n^m(z)$ is the associated Legendre function. Odd spherical harmonics do not appear because $N(\Theta, \Phi)$ is "symmetric" with respect to the origin, i.e. $N(\Theta, \Phi) = N(\pi - \Theta, \Phi + \pi)$. This expression can also be expressed as a Cartesian tensor equation as follows (*cf.* [2]):

$$N(\mathbf{m}) = \frac{C}{4\pi}[1 + D_{ij}m_im_j + D_{ijkl}m_im_jm_km_l + \cdots]. \tag{3.4}$$

Here, $\mathbf{m} = (\sin\Theta\cos\Phi, \sin\Theta\sin\Phi, \cos\Theta)^T$, and the nth fabric tensor $D_{i_1\cdots i_n}$ is expressed in terms of the nth spherical harmonics expansion coefficients A_{nm} and B_{nm}, $m = 1, 2, \ldots, n$ (*cf.* [2]). For example, D_{ij} is given by

$$(D_{ij}) = \begin{bmatrix} -\frac{1}{4}A_{20} + 3A_{22} & 3B_{22} & \frac{3}{2}A_{21} \\ 3B_{22} & -\frac{1}{4}A_{20} - 3A_{22} & \frac{3}{2}B_{21} \\ \frac{3}{2}A_{21} & \frac{3}{2}B_{21} & \frac{1}{2}A_{20} \end{bmatrix}. \tag{3.5}$$

The distribution density $f(\theta, \phi)$ of lines and curves is defined in such a way that $f(\theta, \phi)\sin\theta d\theta d\phi$ is the total length in unit volume of those line elements whose spherical coordinates are between θ and $\theta + d\theta$ and between ϕ and $\phi + d\phi$, where, as before, the orientation of a line element is chosen randomly from the two possibilities with a probability of $\frac{1}{2}$. Thus, $f(\theta, \phi)$ is "symmetric" with respect to the origin, i.e. $f(\theta, \phi) = f(\pi - \theta, \phi + \pi)$, and $c = \int_0^{2\pi}\int_0^{\pi} f(\theta, \phi)\sin\theta d\theta d\phi$ is the total length in unit volume of lines and curves, or the "length density." The distribution density can also be expressed as the following spherical harmonics expansion (see [1] for the proof):

$$f(\theta, \phi) = \frac{C/2\pi}{4\pi}[1 + \sum_{n=2}^{\infty}{}' \lambda_n\{\tfrac{1}{2}A_{n0}P_n(\cos\theta) + \sum_{m=1}^{n} P_n^m(\cos\theta)(A_{nm}\cos m\phi + B_{nm}\sin m\phi)\}]. \tag{3.6}$$

$$\lambda_n = (-1)^{n/2-1}2^{n-1}(n-1)(n+2)\Big/\binom{n}{n/2}. \tag{3.7}$$

In terms of fabric tensors, this can be written as

$$f(\mathbf{n}) = \frac{C/2\pi}{4\pi}[1 + \sum_{n=2}^{\infty}{}' \lambda_n D_{i_1\cdots i_n}n_{i_1}\cdots n_{i_n}] = \frac{C/2\pi}{4\pi}[1 + 4D_{ij}n_in_j - 24D_{ijkl}n_in_jn_kn_l + \cdots], \tag{3.8}$$

where $\mathbf{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)^T$. The equivalent strain is given by

$$e_{ij} = D_{ij}. \tag{3.9}$$

(See [1] for the proof.) Thus, if we compute the spherical harmonics expansion coefficients C, A_{nm} and B_{nm}, $m = 1, 2, \ldots, n$, $n = 0, 1, 2, \ldots$, or equivalently fabric tensors $D_{i_1\ldots i_n}$'s, we know the distribution density completely and the equivalent strain as well. The Monte Carlo method can be adopted to serve that purpose as follows.

Procedure 2

Step 1. Choose N points on the unit sphere randomly according to the uniform distribution and let (Θ_k, Φ_k), $k = 0, 1, 2, \ldots, N-1$ be their spherical coordinates.

Step 2. Cut the material with equally spaced parallel planes whose normal is (Θ_k, Φ_k) in spherical coordinates and let N_k be the number of intersections per unit area of the cutting planes with the lines and curves.

Step 3. Approximate the integrals of eqns (3.2) and (3.3) by appropriate weighted sums of the data. For example, C, A_{20}, A_{21}, B_{21}, A_{22} and B_{22} are given by

$$C = 4\pi \sum_{k=0}^{N-1} N_k/N, \tag{3.10}$$

$$A_{20} = \tfrac{5}{4} \sum_{k=0}^{N-1} N_k(1 + 3\cos 2\Theta_k) \Big/ \sum_{k=0}^{N-1} N_k, \tag{3.11}$$

$$\begin{bmatrix} A_{21} \\ B_{21} \end{bmatrix} = \tfrac{5}{2} \sum_{k=0}^{N-1} N_k \sin 2\Theta_k \begin{bmatrix} \cos\Phi_k \\ \sin\Phi_k \end{bmatrix} \Big/ \sum_{k=0}^{N-1} N_k, \tag{3.12}$$

$$\begin{bmatrix} A_{22} \\ B_{22} \end{bmatrix} = \tfrac{5}{8} \sum_{k=0}^{N-1} N_k(1 - \cos 2\Theta_k) \begin{bmatrix} \cos 2\Phi_k \\ \sin 2\Phi_k \end{bmatrix} \Big/ \sum_{k=0}^{N-1} N_k. \tag{3.13}$$

Step 4. The length density is given by $c = C/2\pi$, and the fabric tensor $D_{i_1\ldots i_n}$ is given as in [2]. In particular, D_{ij} is given by eqn (3.5), and the equivalent strain e_{ij} by eqn (3.9).

4. PROBING SURFACES IN THREE DIMENSIONS WITH LINES

Suppose surfaces are distributed in a three-dimensional material as disjoint fragments like crack surfaces or connected cell walls like grain boundaries or both. Place in the material randomly a line whose orientation is (Θ, Φ) in spherical coordinates, and let $N(\Theta, \Phi)$ be the expected number of intersections per unit length of the probe line with the surfaces. Again, this function is "symmetric" with respect to the origin, i.e. $N(\Theta, \Phi) = N(\pi - \Theta, \Phi + \pi)$, and is expanded into a spherical harmonics expansion of the form of eqn (3.1) or as a Cartesian tensor equation of the form of eqn (3.4), and D_{ij}, for example, is given by eqn (3.5).

The distribution density $f(\theta, \phi)$ of the structure is defined in such a way that $f(\theta, \phi) \times \sin\theta d\theta d\phi$ is the total area in unit volume of those surface elements whose normals lie, in spherical coordinates, between θ and $\theta + d\theta$ and between ϕ and $\phi + d\phi$, where the normal is chosen for each surface element randomly from the two possibilities with a probability of $\frac{1}{2}$. Thus, $f(\theta, \phi)$ is "symmetric" with respect to the origin, i.e. $f(\theta, \phi) = f(\pi - \theta, \phi + \pi)$, and $c = \int_0^{2\pi}\int_0^{\pi} f(\theta, \phi)\sin\theta d\theta d\phi$ is the total area in unit volume of the surfaces, or the "area density." The distribution density is given exactly by the spherical harmonics expansion (3.6), (3.7) or in terms of the fabric tensors by eqn (3.8). In other words, the relation between the number of intersections $N(\Theta, \Phi)$ and the distribution density $f(\theta, \phi)$ is exactly the same with that in the case of lines and curves in a three-dimensional material. (See [1] for the proof.) Thus, one has only to read in the previous section "the orientation of the normal to the cutting plane" as "the orientation of the probe line" and "the orientation of a line element" as "the orientation of the normal to a surface element."

The only thing that differs is that the equivalent strain is given, instead of eqn (3.9), by

$$e_{ij} = -D_{ij}. \tag{4.1}$$

(See [1] for the proof.) If we want to determine the fabric tensors and the equivalent strain by the Monte Carlo Method, the procedure goes similarly.

Procedure 3

Step 1. Choose N points on the unit sphere randomly according to the uniform distribution and let (Θ_k, Φ_k), $k = 0, 1, 2, \ldots, N - 1$, be their spherical coordinates.

Step 2. Cut the material with equally spaced parallel planes on which lies the orientation (Θ_k, Φ_k) and draw on each cross-section parallel lines of orientation (Θ_k, Φ_k) whose spacing is the same as that of the parallel planes. Let N_k be the number of intersections per unit length of the probe lines with the surfaces.

Step 3. Compute the spherical harmonics expansion coefficients as before. In particular, C, A_{20}, A_{21}, B_{21}, A_{22}, and B_{22} are given by eqns (3.10)–(3.12) and hence D_{ij} by eqn (3.5).

Step 4. The area density is given by $c = C/2\pi$ and the fabric tensors are given as before. In particular, D_{ij} is given by eqn (3.5), and the equivalent strain e_{ij} by eqn (4.1).

5. PROBING SURFACES IN THREE DIMENSIONS WITH PLANES

Suppose, as in the previous section, that surfaces are distributed in a three-dimensional material. This time, instead of placing a line in the material, cut the material randomly with a plane whose normal is in spherical coordinates (Θ, Φ) and let $N(\Theta, \Phi)$ be the expected value of the total "length" of the intersection curves that appear on unit area of the cross-section. Again, $N(\Theta, \Phi)$ is "symmetric" with respect to the origin, i.e. $N(\Theta, \Phi) = N(\pi - \Theta, \Phi + \pi)$, and is expanded into the spherical harmonics expansion of eqn (3.1) or as a Cartesian tensor equation of eqn (3.4). In particular, D_{ij} is given by eqn (3.5). In this case, the distribution density $f(\theta, \phi)$ is given by

$$f(\theta, \phi) = \frac{C/\pi^2}{4\pi}\Big[1 + \sum_{n=2}^{\infty}{}' \mu_n\Big\{\tfrac{1}{2}A_{20}P_n(\cos\theta) + \sum_{m=1}^{n} P_n^m(\cos\theta)(A_{nm}\cos m\phi + B_{nm}\sin m\phi)\Big\}\Big]. \tag{5.1}$$

$$\mu_n = -2^{2n-1}(n-1)/n\binom{n}{n/2}^2. \tag{5.2}$$

(See [1] for the proof.) In terms of fabric tensors, this becomes

$$f(\mathbf{n}) = \frac{C/\pi^2}{4\pi}\Big[1 + \sum_{n=2}^{\infty}{}' \mu_n D_{i_1\cdots i_n} n_{i_1}\cdots n_{i_n}\Big]$$

$$= \frac{C/\pi^2}{4\pi}\Big[1 - D_{ij}n_in_j - \tfrac{8}{3}D_{ijkl}n_in_jn_kn_l + \cdots\Big]. \tag{5.3}$$

The equivalent strain is given by

$$e_{ij} = \tfrac{1}{4}D_{ij}. \tag{5.4}$$

(See [1] for the proof.) The Monte Carlo method goes as follows.

Procedure 4

Step 1. Choose N points on the unit sphere randomly according to the uniform distribution and let (Θ_k, Φ_k), $k = 0, 1, 2, \ldots, n - 1$, be their spherical coordinates.

Step 2. Cut the material with equally spaced parallel planes whose normal is (Θ_k, Φ_k) in spherical coordinates and let N_k be the total length of the intersection curves per unit area that appear on the cross-sections.

Step 3. Compute the spherical harmonics expansion coefficients as before. In particular, C, A_{20}, A_{21}, B_{21}, A_{22}, and B_{22} are given by eqns (3.10)–(3.13).

Step 4. The area density is given by $c = C/\pi^2$ and the fabric tensors are given as before. In particular, D_{ij} is given by eqn (3.5), and the equivalent strain e_{ij} by eqn (5.4).

6. DETERMINATION OF ANISOTROPY BY RESTRICTED OBSERVATIONS

The Monte Carlo method described so far requires the generation of random numbers, and we must cut the material with planes of many different orientations. In order to do so, we must prepare as many material samples all of which are supposed to have statistically the same structural anisotropy. This may be a serious obstacle in some situations. Is it not possible to determine the anisotropy by cutting the material with planes of special orientations like those parallel to coordinate planes? This is not possible, in general, because the information obtained is insufficient. The Monte Carlo method is the most consistent way, in the sense that we can determine the spherical harmonics expansion coefficients of any degree "in principle," i.e. if the number N of observations is sufficiently large and the spacing of parallel lines or planes is sufficiently small. However, determination by restricted observations is possible only if the spherical harmonics expansion of the data, and hence that of the distribution density itself, do not have high spherical harmonics above a certain degree. The simplest solution of this situation is when the distribution density has only zeroth and second degree terms. In this case, we say that the anisotropy is "weak." This type of distribution has the symmetry of D_{2h} (in the Schönflies notation), a special case of orthogonal anisotropy.

If the anisotropy is weak, the distribution of observed data, i.e. either the "number" or the "length" of intersections with the probe, is written in the form of

$$N(\mathbf{m}) = \frac{C}{4\pi}[1 + D_{ij}m_im_j]. \tag{6.1}$$

As has been shown so far, our goal of determining the distribution density $f(\mathbf{n})$ and the equivalent strain e_{ij} is attained by determining the coefficient C and the fabric tensor D_{ij} from the data $N(\mathbf{m})$ of various $\mathbf{m}$. Since D_{ij} is a symmetric deviator tensor, i.e. $D_{ij} = D_{ji}$ and $D_{ii} = 0$ (*cf.* [2]), it has only five independent elements. Hence, it is possible in principle to determine C and D_{ij} from observations of six different orientations of $\mathbf{m}$. However, that would produce an unreliable result, because the result is sensitively vulnerable to possible errors or fluctuations of the data. It is, therefore, preferable to obtain the data in the form of sums or averages of a large number of observations, yet such that the observations are restricted to special types.

Let us first consider the case where we are to count the number of intersections with probe lines. If we cut a material with a plane whose unit normal is $\mathbf{l}$, we can draw, on the cross-section, lines of any orientation orthogonal to $\mathbf{l}$. Suppose the orientation of the probe line is restricted to be orthogonal to a fixed orientation $\mathbf{l}$. Consider the following quantities

$$M(\mathbf{l}) = \int_{C(\mathbf{l})} N(\mathbf{m})\mathrm{d}\mathbf{m}, \tag{6.2}$$

$$M_{ij}(\mathbf{l}) = \int_{C(\mathbf{l})} m_im_jN(\mathbf{m})\mathrm{d}\mathbf{m}. \tag{6.3}$$

Here, $C(\mathbf{l})$ is the unit circle encircling $\mathbf{l}$ perpendicularly and $\mathrm{d}\mathbf{m}$ is the uniform measure of $C(\mathbf{l})$ normalized to 2π, i.e. the differential azimuthal angle with $\mathbf{l}$ as the pole (Fig. 4). Since $\mathbf{m}$ is a unit vector, we see that $M(\mathbf{l}) = M_{ii}(\mathbf{l})$. If we substitute eqn (6.1) in eqns (6.2) and (6.3), we obtain

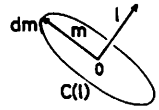

Fig. 4. Unit vector m is on circle $C(\mathbf{l})$ encircling unit vector l perpendicularly.

$$M(\mathbf{l}) = \frac{C}{2}(1 - \tfrac{1}{2}D_{ij}l_il_j), \tag{6.4}$$

$$M_{ij}(\mathbf{l}) = \frac{C}{4}[\tfrac{1}{2}D_{ij} - D_{k(i}l_{j)}l_k + (1 - \tfrac{1}{4}D_{kl}l_kl_l)\delta_{ij} - (1 - \tfrac{4}{3}D_{kl}l_kl_l)l_il_j], \tag{6.5}$$

where δ_{ij} is the Kronecker delta. These are easily obtained if we note the identities

$$\int_{C(\mathbf{l})} \mathrm{d}\mathbf{m} = 2\pi, \tag{6.6}$$

$$\int_{C(\mathbf{l})} m_im_j\mathrm{d}\mathbf{m} = \pi(\delta_{ij} - l_il_j), \tag{6.7}$$

$$\int_{C(\mathbf{l})} m_im_jm_km_l\mathrm{d}\mathbf{m} = \tfrac{3}{4}\pi(\delta_{(ij}\delta_{kl)} - 2l_{(i}l_j\delta_{kl)} + l_il_jl_kl_l), \tag{6.8}$$

where ( ) designates the symmetrization of indices. Their validity is directly checked if we take such a Cartesian coordinate system that $\mathbf{l} = (0, 0, 1)^{\mathrm{T}}$. Since they are Cartesian tensor equations invariant to coordinate rotations, they are necessarily valid in any coordinate system once they are valid in some coordinate system.

Let us fix a Cartesian coordinate system and let $\mathbf{e}_1 = (1, 0, 0)^{\mathrm{T}}$, $\mathbf{e}_2 = (0, 1, 0)^{\mathrm{T}}$ and $\mathbf{e}_3 = (0, 0, 1)^{\mathrm{T}}$ be the basis vectors. From eqn (6.4), we see that

$$M(\mathbf{e}_i) = \frac{C}{2}(1 - \tfrac{1}{2}D_{ii}), \qquad (i \text{ not summed}). \tag{6.9}$$

Adding this for $i = 1, 2, 3$, and noting $D_{ii} = 0$ (i summed), we obtain an expression for C in the form

$$C = \tfrac{2}{3}[M(\mathbf{e}_1) + M(\mathbf{e}_2) + M(\mathbf{e}_3)]. \tag{6.10}$$

Then, from eqn (6.9), we obtain expressions for D_{11}, D_{22}, and D_{33} as

$$D_{11} = 2[-2M(\mathbf{e}_1) + M(\mathbf{e}_2) + M(\mathbf{e}_3)]/[M(\mathbf{e}_1) + M(\mathbf{e}_2) + M(\mathbf{e}_3)], \tag{6.11}$$

$$D_{22} = 2[M(\mathbf{e}_1) - 2M(\mathbf{e}_2) + M(\mathbf{e}_3)]/[M(\mathbf{e}_1) + M(\mathbf{e}_2) + M(\mathbf{e}_3)], \tag{6.12}$$

$$D_{33} = 2[M(\mathbf{e}_1) + M(\mathbf{e}_2) - 2M(\mathbf{e}_3)]/[M(\mathbf{e}_1) + M(\mathbf{e}_2) + M(\mathbf{e}_3)]. \tag{6.13}$$

If i, j, k is a permutation of 1, 2, 3, we obtain from eqn (6.5)

$$M_{ij}(\mathbf{e}_k) = \frac{C}{8}D_{ij}. \tag{6.14}$$

From this and eqn (6.10), we obtain expressions for D_{12}, D_{23}, and D_{31} as

$$D_{12} = 12M_{12}(\mathbf{e}_3)/[M(\mathbf{e}_1) + M(\mathbf{e}_2) + M(\mathbf{e}_3)], \tag{6.15}$$

$$D_{23} = 12M_{23}(\mathbf{e}_1)/[M(\mathbf{e}_1) + M(\mathbf{e}_2) + M(\mathbf{e}_3)], \tag{6.16}$$

$$D_{31} = 12M_{31}(\mathbf{e}_2)/[M(\mathbf{e}_1) + M(\mathbf{e}_2) + M(\mathbf{e}_3)]. \tag{6.17}$$

If the distribution is nearly but not strictly weak, the above results give approximations of C and D_{ij}. A rough estimation of involved error is obtained by considering in eqn (6.1) the next term of D_{ijkl}. Then, the right-hand side of eqn (6.9) becomes $(C/2)(1 - (\frac{1}{2})D_{ij} + (\frac{3}{8})D_{iiii})$ (i not summed). Hence, the right-hand side of eqn (6.10)becomes $C(1 + (\frac{1}{8}) \sum_i D_{iiii})$, i.e. the estimate of C by eqn (6.10) involves relative error $(\frac{1}{8}) \sum_i D_{iiii}$. Similarly, the right-hand side of eqn (6.14) becomes $(C/8)(D_{ij} - D_{ijkk})$ (k not summed), and the estimate of D_{ij} by the right-hand sides of eqns (6.15)–(6.17) becomes $(D_{ij} - D_{ijkk})/(1 + (\frac{1}{8}) \sum_l D_{llll})$, where i, j, k is a permutation of 1, 2, 3 and k is not summed.

7. PROCEDURES WITH RESTRICTED PROBE LINES AND PLANES

Consider the case of counting the number of intersections on probe lines with surfaces in the material. The procedure described in the previous section is summarized as follows.

Procedure 5

Step 1. Fix a Cartesian coordinate system in the material. Cut the material with equally spaced planes parallel to the ij-plane.

Step 2. On each cross-section, draw equally spaced parallel lines whose spacing is equal to that of the cutting planes and whose orientation is $m\pi/N$, $m = 0, 1, 2, \ldots, N - 1$, from the i-axis (or from the j-axis). Let $M_m^{(ij)}$ be the number of intersections per unit length of the probe lines of orientation $m\pi/N$ (see Fig. 5).

Step 3. Compute $M(\mathbf{e}_k)$ and $M_{ij}(\mathbf{e}_k)$ by

$$M(\mathbf{e}_k) = 2\pi \sum_{m=0}^{N-1} N_m^{(ij)}/N, \tag{7.1}$$

$$M_{ij}(\mathbf{e}_k) = \pi \sum_{m=0}^{N-1} N_m^{(ij)} \sin (2\pi m/N)/N, \tag{7.2}$$

where i, j, k is a permutation of 1, 2, 3.

Step 4. All necessary data are obtained by doing this for the xy-, yz-, and zx-planes. The coefficient C is given by eqn (6.10) and, the fabric tensor D_{ij} by eqns (6.11)–(6.13), (6.15)–(6.17). The length or area density c, the distribution density $f(\mathbf{n})$, and the equivalent strain e_{ij} are obtained from C and D_{ij} as stated before.

Thus, we have only to cut the material with planes parallel to the coordinate planes and hence to prepare just three material samples supposedly of statistically the same anisotropy. The necessary data are obtained by "planning down" a plane surface successively.

Next, consider the case of counting the number of points or measuring the length of curves on cutting planes. This time, only one data is obtained for one orientation of the cutting plane, and hence cutting with planes parallel to the coordinate planes is not sufficient. Still, there exists a method which is a kind of "dual" to the above described method, requiring only three material samples. As before, let i, j, k be a permutation of 1, 2, 3. Suppose the distribution of the internal structure is spatially homogeneous.

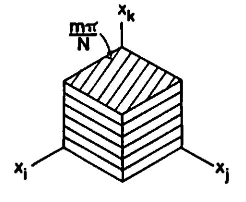

Fig. 5. Equally spaced planes parallel to the ij-plane cutting a material. Observations are made on each plane.

Procedure 6

Step 1. Cut the material with equally spaced "concentric cylindrical surfaces" whose central axis coincides with the k-axis.

Step 2. Consider, on each cylindrical surface, the strip defined by $2\pi(m - 1/2)/N \leq \phi < 2\pi(m + 1/2)/N$, where ϕ is the angle from the i-axis (or from the j-axis). (See Fig. 6). Observe that strip (i.e. count the number of points or measure the length of curves on it), and divide the observed data by the area of that strip. Let $N_m^{(ij)}$, $m = 0, 1, 2, \ldots, N - 1$, be the "weighted average" of that value over all the cylinders, the weight being proportional to the radius of the cylinder.

Step 3. Compute $M(\mathbf{e}_i)$ and $M_{ij}(\mathbf{e}_k)$ by

$$M(\mathbf{e}_k) = 2\pi \sum_{m=0}^{N-1} N_m^{(k)}/N, \tag{7.3}$$

$$N_{ij}(\mathbf{e}_k) = \pi \sum_{m=0}^{N-1} N_m^{(k)} \sin(4\pi m/N)/N. \tag{7.4}$$

Step 4. All necessary data are obtained by doing this for cylinders with the x-, y-, and z-axes as the axes. The coefficient C is given by eqn (6.10), and the fabric tensor D_{ij} by eqns (6.11)–(6.13), (6.15)–(6.17). The length or area density c, the distribution density $f(\mathbf{n})$ and the equivalent strain e_{ij} are obtained as before.

Thus, we need to prepare just three material samples supposedly of statistically the same anisotropy. We can obtain the necessary data by "lathing" a big cylinder, thinning it successively.

8. CONCLUDING REMARKS

In this article, we have presented practical procedures of determining the structural anisotropy and the equivalent strain by the stereological method, i.e. by counting the number of points on probe lines or cutting planes or by measuring the length of curves on cutting planes, on the basis of the theoretical study in the previous articles [1, 2]. First, we illustrated the relationship between the observed data and the distribution density of the internal structure in terms of both polar or spherical coordinates and Cartesian tensors. We also gave the form of the equivalent strain tensor explicitly.

We then described an actual procedure for the two-dimensional case. It is quite straightforward, and we gave a synthetic example. However, things are not so simple in the three-dimensional case. First, we described the Monte Carlo method which computes all relevant quantities. This is theoretically the most consistent method. However, it requires a large number of material samples supposedly of statistically the same anisotropy. Then, we gave procedures requiring only three material samples, assuming that the structural anisotropy is "weak," i.e. the distribution density has only spherical harmonics up to degree 2. We have only to cut the material with planes parallel to the coordinate planes or cylindrical surfaces around the coordinate axes. We also gave an estimate of involved error of approximation when the distribution is nearly weak.

In this article, we used equally spaced parallel lines or planes both in the Monte Carlo method and in the restricted cutting. This is because we did not assume the "translational homogeneity" of the internal structure of the material, so that we must place lines or

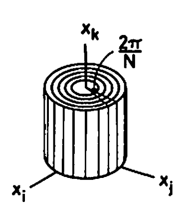

Fig. 6. Equally spaced concentric cylinders along the k-axis cutting a material. Observations are made on each cylinder.

planes "homogeneously" according to the invariant Haar measure [8]. On the other hand, the use of concentric cylinders is possible only when the distribution is homogeneous. If the distribution is known to be roughly translationally homogeneous and uncorrelated between different points, we need not use equally spaced lines or planes. In fact, only one representative line or plane in the material can be used for each cutting orientation. However, the variance of the data increases inversely proportional to the total length or area of the probe line or the cutting plane.

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