Proc. 10th RCIM Symp. Math. Aspects Image Process. Comput. Vis., November 2010, Sapporo, Japan, pp. 3–4.

Hyper Least Squares and Its Applications

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1. Introduction

A fundamental problem in computer vision is the extraction of 2-D/3-D geometric information from noisy observations, for which the maximum likelihood (ML) estimator is known to provide a highly accurate solution [3, 4]. However, ML computation is usually iterative and may not converge for high noise levels. The convergence critically depends on the accuracy of the initial guess to start the iterations, for which the least squares (LS) estimator has been widely used with limited accuracy. Following Kanatani [5], we do rigorous error analysis and derive a new LS estimator called "hyperLS" with accuracy comparable to ML. The improved accuracy results from introduction of a normalization that eliminates the statistical bias up to second order noise terms.

2. Geometric Fitting

Suppose noisy observations $\boldsymbol{x}_1$, ..., $\boldsymbol{x}_N$ are perturbations in the true values $\bar{\boldsymbol{x}}_1$, ..., $\bar{\boldsymbol{x}}_N$ that satisfy implicit geometric constraints of the form

$$F^{(k)}(\boldsymbol{x};\boldsymbol{\theta}) = 0, \qquad k = 1, ..., L. \tag{1}$$

The unknown parameter $\boldsymbol{\theta}$ allows us to infer the 2-D/3-D shape and motion of the observed objects [3, 4]. We call problems of this type *geometric fitting*. In many important applications, the problem can be reparameterized to make the functions $F^{(k)}(\boldsymbol{x};\boldsymbol{\theta})$ linear in $\boldsymbol{\theta}$ (but nonlinear in $\boldsymbol{x}$) so that Eq. (1) is written as

$$(\boldsymbol{\xi}^{(k)}(\boldsymbol{x}), \boldsymbol{\theta}) = 0, \qquad k = 1, ..., L, \tag{2}$$

where and hereafter $(\boldsymbol{a}, \boldsymbol{b})$ denotes the inner product of vectors $\boldsymbol{a}$ and $\boldsymbol{b}$. The vector $\boldsymbol{\xi}^{(k)}(\boldsymbol{x})$ represents a nonlinear mapping of $\boldsymbol{x}$.

3. Algebraic Methods

We abbreviate $\boldsymbol{\xi}^{(k)}(\boldsymbol{x}_\alpha)$ as $\boldsymbol{\xi}_\alpha^{(k)}$. *Algebraic* methods refer to those minimizing the *algebraic distance*

$$J = \frac{1}{N}\sum_{\alpha=1}^{N}\sum_{k=1}^{L}(\boldsymbol{\xi}_\alpha^{(k)}, \boldsymbol{\theta})^2 = \frac{1}{N}\sum_{\alpha=1}^{N}\sum_{k=1}^{L}\boldsymbol{\theta}^\top\boldsymbol{\xi}_\alpha^{(k)}\boldsymbol{\xi}_\alpha^{(k)\top}\boldsymbol{\theta} = (\boldsymbol{\theta}, \boldsymbol{M}\boldsymbol{\theta}), \tag{3}$$

where we define

$$\boldsymbol{M} = \frac{1}{N}\sum_{\alpha=1}^{N}\sum_{k=1}^{L}\boldsymbol{\xi}_\alpha^{(k)}\boldsymbol{\xi}_\alpha^{(k)\top}. \tag{4}$$

Equation (3) is trivially minimized by $\boldsymbol{\theta} = \boldsymbol{0}$ unless scale normalization is imposed on $\boldsymbol{\theta}$. The most common normalization is $\|\boldsymbol{\theta}\| = 1$; we call this the *standard LS*. However, *the solution depends on the normalization*. The aim of this paper is to find a normalization that maximizes the accuracy of the solution. This issue has been raised by Al-Sharadqah and Chernov [1] and Rangarajan and Kanatani [8] for circle fitting, by Kanatani and Rangarajan [6] for ellipse fitting, and by Niitsuma et al. [7] for homography estimation. In this work, we generalize their results to an arbitrary number of constraints in Eq. (2). Following [1, 6, 7, 8], we consider the class of normalizations

$$(\boldsymbol{\theta}, \boldsymbol{N}\boldsymbol{\theta}) = \text{constant}. \tag{5}$$

This approach can be regarded as an extension of the well known method of Taubin [10], but we show that our result exceeds it.

Traditionally, the matrix $\boldsymbol{N}$ is assumed to be positive definite, but here we allow nondefinite (i.e., neither positive nor negative definite) matrices and search for $\boldsymbol{N}$ that maximizes the accuracy. For such an $\boldsymbol{N}$, the solution that minimizes Eq. (3) subject to Eq. (5), if it exists, is obtained by solving the generalized eigenvalue problem

$$\boldsymbol{M}\boldsymbol{\theta} = \lambda\boldsymbol{N}\boldsymbol{\theta}. \tag{6}$$

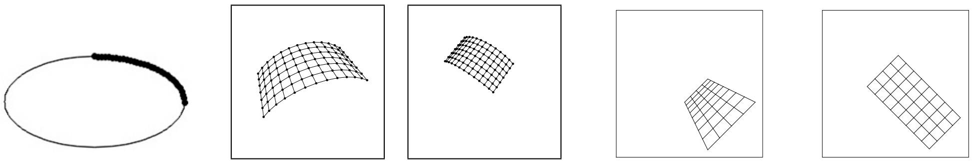
Evidently, $\lambda = 0$ in the absence of noise. If $\boldsymbol{N}$ is positive definite, the parameter $\boldsymbol{\theta}$ is estimated as the generalized eigenvector for the smallest eigenvalue λ, but in other cases for the smallest absolute value $|\lambda|$. To avoid the possibility that the expectation of θ diverges to ∞, we regard Eq. (6) as the definition of our "algebraic method". Then, the solution $\boldsymbol{\theta}$ can be normalized to $\|\boldsymbol{\theta}\| = 1$ rather than Eq. (5).

4. HyperLS

We propose to use as $\boldsymbol{N}$

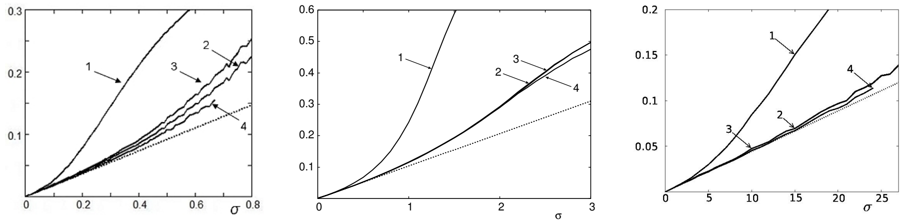
$$\boldsymbol{N} = \boldsymbol{N}_{\rm T} - \frac{1}{N^2}\sum_{\alpha=1}^{N}\sum_{k,l=1}^{L}\Big(\mathrm{tr}[\boldsymbol{M}^- V^{(kl)}[\boldsymbol{\xi}_\alpha]]\boldsymbol{\xi}_\alpha^{(k)}\boldsymbol{\xi}_\alpha^{(l)\top} + (\boldsymbol{\xi}_\alpha^{(k)}, \boldsymbol{M}^-\boldsymbol{\xi}_\alpha^{(l)})V^{(kl)}[\boldsymbol{\xi}_\alpha] + 2\mathcal{S}[V^{(kl)}[\boldsymbol{\xi}_\alpha]\boldsymbol{M}^-\boldsymbol{\xi}_\alpha^{(k)}\boldsymbol{\xi}_\alpha^{(l)\top}]\Big), \tag{7}$$

where $\mathcal{S}[\,\cdot\,]$ denotes symmetrization ($\mathcal{S}[\boldsymbol{A}] = (\boldsymbol{A} + \boldsymbol{A}^\top)/2$). The superscript $(\,\cdot\,)^-$ denotes the Moore-Penrose pseudoinverse. The matrix $\boldsymbol{N}_{\rm T}$ is defined to



(a) (b) (c)

Figure 1: (a) 31 points on an ellipse. (b) Two views of a curved grid. (c) Two views of a planar grid.



(a) (b) (c)

Figure 2: RMS error vs. the standard deviation σ of noise added to each point. 1. standard LS, 2. hyperLS, 3. Taubin approximation, 4. ML. The dotted lines indicate the KCR lower bound. (a) Ellipse fitting. (b) Fundamental matrix computation. (c) Homography computation.

be

$$\boldsymbol{N}_{\mathrm{T}} = \frac{1}{N}\sum_{\alpha=1}^{N}\sum_{k=1}^{L}\Big(V^{(kk)}[\boldsymbol{\xi}_\alpha] + 2\mathcal{S}[\boldsymbol{\xi}_\alpha^{(k)}\boldsymbol{e}_\alpha^{(k)\top}]\Big), \quad (8)$$

where $\boldsymbol{e}_\alpha^{(k)}$ is the expectation of the second order error in $\Delta\boldsymbol{\xi}_\alpha$. The covariance of $\boldsymbol{\xi}_\alpha^{(k)}$, $k = 1, ..., L$, is defined by by ($E[\,\cdot\,]$ denotes expectation)

$$\begin{aligned} V^{(kl)}[\boldsymbol{\xi}_\alpha] &\equiv E[\Delta_1\boldsymbol{\xi}_\alpha^{(k)}\Delta_1\boldsymbol{\xi}_\alpha^{(l)\top}] \\ &= \boldsymbol{T}_\alpha^{(k)}E[\Delta\boldsymbol{x}_\alpha\Delta\boldsymbol{x}_\alpha^\top]\boldsymbol{T}_\alpha^{(l)\top} = \boldsymbol{T}_\alpha^{(k)}V[\boldsymbol{x}_\alpha]\boldsymbol{T}_\alpha^{(l)\top}, \quad (9) \end{aligned}$$

where $V[\boldsymbol{x}_\alpha]$ is the covariance matrix of observation $\boldsymbol{x}_\alpha$, and $\boldsymbol{T}_\alpha^{(k)}$ is the Jacobian matrix defined by

$$\boldsymbol{T}_\alpha^{(k)} \equiv \left.\frac{\partial\boldsymbol{\xi}^{(k)}(\boldsymbol{x})}{\partial\boldsymbol{x}}\right|_{\boldsymbol{x}=\bar{\boldsymbol{x}}_\alpha}. \quad (10)$$

Standard linear algebra routines for solving the generalized eigenvalue problem of Eq. (6) assume that $\boldsymbol{N}$ is positive definite, but here $\boldsymbol{N}$ is nondefinite. We circumvent this problem by rewriting (6) in the form

$$\boldsymbol{N}\boldsymbol{\theta} = \frac{1}{\lambda}\boldsymbol{M}\boldsymbol{\theta}. \quad (11)$$

The matrix $\boldsymbol{M}$ in Eq. (4) is positive definite except in the absence of noise, in which case the smallest eigenvalue is 0.

5. Numerical Experiments

We fit an ellipse to the point sequence shown in Fig. 1(a). We compute the fundamental matrix between the two images shown in Fig. 1(b). We compute the homography relating the two images shown in Fig. 1(c).

Figure 2 plots for the noise level σ, the RMS error for each method, the Taubin approximation [9], and the theoretical accuracy limit called the *KCR lower bound* [2, 4, 5]. In all examples, the standard LS performs poorly, while ML provides the highest accuracy. We also see that ML computation fails in the presence of large noise. In contrast, hyperLS can produce a solution close to ML in any noise level.

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