

A PLASTICITY THEORY FOR THE KINEMATICS OF IDEAL GRANULAR MATERIALS

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Abstract—A plasticity theory is formulated for ideal granular materials. The material is assumed to be isotropic and incompressible. All the equations are expressed in the form of 3-dimensional tensor equations. This theory is based on a new interpretation of the associated flow rule with associated constraints of deformation. The characteristic surface is analyzed to see the possible discontinuity of the deformation. Then the elastic strain is incorporated and the singular wave propagation is analyzed to determine the possible velocities and directions of propagation.

1. INTRODUCTION

IT HAS BEEN SHOWN that the statics of granular materials such as soils and powders based on the Coulomb criterion provides a satisfactory basis for the analysis of limit equilibrium (e.g. Sokolovskii[1]). However, various attempts to describe deformations have been less successful for further development. In fact, there exist many different schools of thought, each claiming a particular type of formalism. This reflects the fact that the range of types of granular material and their behaviour is indeed very great. Therefore, any mathematical model is a drastic idealization of actual materials and should not be expected to be valid over a wide range of conditions.

The first attempt to give a theoretical foundation on the basis of the plasticity theory may be attributed to Drucker and Prager[2]. However, they have often been criticized later. The issue is twofold. First, their theory predicted an increase in specific volume to an unrealistic extent during shear deformations. The second point is closely related to the former. They applied the associated flow rule, which had been developed for metal plasticity, but it has been argued that the rule does not hold when applied to granular materials in which frictional stresses are involved. For metals, the rule is implied either by the Bishop-Hill theory of polycrystals[3] or by the plastic work postulate of Drucker[4] or Il'iushin[5]. It has been pointed out, however, that neither holds for granular materials (e.g. Mandel[6]). It has also been shown that the existence of the plastic potential or the associated flow rule is expected only when the microscopic *rearrangement* of the material is governed via the thermodynamic *conjugate* force (Rice[7, 8]). Therefore, the associated flow rule does not necessarily follow in granular materials in which the microscopic frictional slip is the mechanism of deformation.

Since the work of Drucker and Prager[2], the subsequent theories can be classified into two categories according to the effort to overcome one of the two aforesaid defects. As for the former, various types of yield functions, plastic potentials distinct from the yield functions and hardening rules have been elaborated for the fitting of experimental data[9-12]. This approach has the merit that the basic principle is simple and easy to manipulate. Moreover, the mechanical foundation need not be taken seriously in practical applications. Instead, however, the equations tend to be complex and lengthy, and a number of *ad hoc* assumptions become necessary.

As for the second point, on the other hand, efforts have been made to pursue a mechanically consistent theory not relying on the dubious associated flow rule. Spencer[13] introduced the idea that deformation occurs by shear on certain critical planes, assuming that the strain-rate is dependent on the stress-rate as well as the stress. Goodman and Cowin[14] assumed that the stress depends on the gradient of the solid volume fraction of the material as well as on the strain-rate. Kanatani[15] analyzed microscopic interparticle friction and collisions and took statistical average of the interactions to obtain constitutive equations for an equivalent model of the flow. He also analyzed the distribution of the voidage and the contact forces by a statistical consideration[16, 17].

In this paper, we reexamine the Drucker–Prager theory. We first review the associated flow rule and observe that the rule need not be abolished if it is used in a different sense. Then, we present a plastic theory for ideal granular materials based on the new interpretation of the associated flow rule. All the equations are expressed in the form of 3-dimensional Cartesian tensor equations. We limit our discussion in small deformation, since the analysis of large deformation requires quantitatively different and highly sophisticated discussions such as the selection of the reference coordinate system [8]. Here, we consider the *stable flow regime* of deformation without dilatancy, hardening and unloading. These effects can be incorporated by the same principle [18]. Our theory for the perfect plasticity turns out a straight extension of both the Levi–Mises theory of metal plasticity and Kanatani's theory of statistical construction [15]. Then, we analyze the characteristic surfaces to see the possible occurrence of discontinuity (Hill [19], Thomas [20]) or physically the *localization* of deformation (Mandel [6], Rudnicki and Rice [21], Stören and Rice [22], see also [23, 24]). Then we incorporate the elastic strain to obtain an extension of the Prandtl–Reuss theory of metal plasticity. Finally, we analyze the propagation of singular surfaces and determine the possible velocities and directions of propagation. Our theory is shown to include the statics of limit equilibrium in it, so that it is a natural extension of the statics to kinematics.

2. THE ASSOCIATED FLOW RULE FOR GRANULAR MATERIALS

The associated flow rule is the rule that relates the plastic strain-rate and the stress through differentiation of the yield function. This procedure was first worked out in the theory of metal plasticity [25]. Later, Drucker [4] proposed what he called the *fundamental postulate of material stability* and derived the rule from his postulate. His postulate states that *when a body is in an arbitrary equilibrium, the work done by any cycle of application-and-removal of additional loading is non-negative*. Let the initial stress in an equilibrium at time $t = 0$ be σ_{ij}^* , and let t_1 designate the first occurrence of plastic strain. Furthermore, let the loading be continued until $t = t_2$ and the removal of the added load take place until $t = t_3$ when the stress is again σ_{ij}^* . Since the work done by the stress on the elastic strain during a closed cycle vanishes, and since plastic deformation occurs only during the interval $t_1 < t < t_2$, the work done by the additional loading is

$$\int_{t_1}^{t_2} (\sigma_{ij} - \sigma_{ij}^*) \dot{\epsilon}_{ij}^p dt, \quad (1)$$

where $\dot{\epsilon}_{ij}^p$ is the plastic strain-rate. Throughout this paper, we adopt the Cartesian tensor notation and the rule of summation convention. Taking the limit $t_2 \rightarrow t_1$ and applying Drucker's postulate to it, we obtain

$$(\sigma_{ij} - \sigma_{ij}^*) \dot{\epsilon}_{ij}^p \geq 0 \quad (2)$$

for an *arbitrary equilibrium stress* σ_{ij}^* . This means that the angle made by the six-dimensional vectors $\sigma_{ij} - \sigma_{ij}^*$ and $\dot{\epsilon}_{ij}^p$ is not greater than $\pi/2$. Then, we can conclude that (i) the yield surface is convex and (ii) if the surface is smooth, vector $\dot{\epsilon}_{ij}^p$ is normal to the yield surface at the point of the yield stress σ_{ij} . Consequently, if the yield equation $f(\sigma_{ij}) = 0$ is regular, then we have for the plastic strain-rate

$$\dot{\epsilon}_{ij}^p = \Lambda \frac{\partial f}{\partial \sigma_{ij}}, \quad (3)$$

where Λ is a scalar quantity. This is also referred to as the *normality condition*.

Later, Il'iushin [5] asserted that the cycle of loading be replaced by a cycle of the total strain. It has been known that although Drucker's postulate and Il'iushin's postulate lead to similar results when only infinitesimal deformations are considered, the former is not invariant to the definition of conjugate stress in the case of finite deformations whereas the latter is (Hill [26], Hill and Rice [27]). Yamamoto [28] and Green and Naghdi [29] gave generalizations which include thermal effects as well. On the other hand, it is well known that the associated

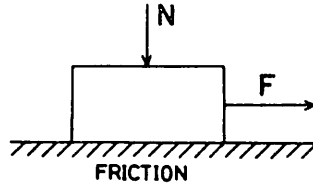


Fig. 1. Friction on a plane.

flow rule predicts an increase in specific volume when applied to granular materials of Coulomb-type yield surfaces. This is easily understood, as was pointed out by Drucker [30] and Mandel [6], if we consider the following example. Imagine a block on a horizontal plane as is shown in Fig. 1. If the friction between the block and the plane obeys the Coulomb law, the yield equation is $F = \pm \mu N$, where μ is the friction coefficient. As is seen from Fig. 2, the associated flow rule predicts normal displacements which do not actually occur. Suppose the block is in equilibrium under the normal force N^* and the horizontal force F^* . Let us apply a force F which has a small backward horizontal component but has an upward normal component large enough to cause slip (Fig. 3). Then, remove the force to reduce the system in the initial state of equilibrium forces. The work done by the added force during this process is clearly negative, because the block moves in the direction opposite to the horizontal component of F . (Note that the normal component of F does not do any work.) Thus, Drucker's postulate is violated, and the system is not stable in the sense of Drucker. It is evident that if the block could move upward, Drucker's postulate would be satisfied.

The frictional slip is a basic concept in the mechanics of granular materials, for the Coulomb criterion is derived from the local slip condition on potential slip-planes. If local slips are the mechanism of deformation, the specific volume must be conserved during deformations. This fact assigns an *internal constraint* of deformation. Of course, the so called dilatancy may occur in granular materials. However, this phenomenon is closely related to the packing configuration of the constituent granules [16], and hence it cannot be expected to be derived from the Coulomb-type criterion alone. In fact, we should not ascribe too many effects to one criterion. Rather, they should be treated separately [18].

Now, we try to modify Drucker's postulate so that it can be applied to plastic deformations with associated internal constraints. A constraint of deformation defines an associated constraining stress, which is a portion of the stress that does not do any work for admissible

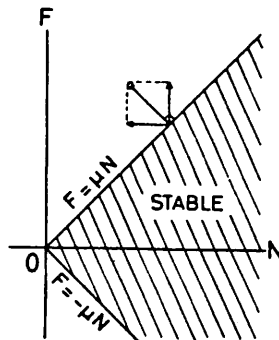


Fig. 2. The yield criterion for friction on a plane.

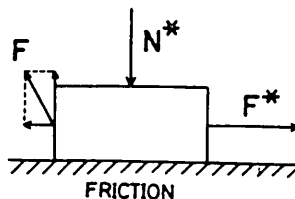


Fig. 3. A counterexample of Drucker's postulate.

deformations but does work only for virtual displacements violating the constraint. Since the constraints give an additional set of kinematic equations, the corresponding constraining stresses must be treated as so many additional independent kinematic variables, which are often referred to as *Lagrange multipliers* of the constraints. Hence, we must treat the constraining stresses separately from the remaining potential and dissipative stresses. Drucker's postulate is apparently a demand for the dissipative stresses. Hence, we propose the following modification to Drucker's postulate; *when a body is in an arbitrary equilibrium, the work done by any cycle of application-and-removal of loading such that the constraining stresses are kept constant is non-negative*. The normal force N in Fig. 1 is the constraining force for slips on the plane, and it is easily seen that this new postulate is satisfied by that system. The constraining stress associated with the constraint of incompressibility is simply the hydrostatic pressure $p = -(1/3)\sigma_{kk}$. Following the previous procedure, we again obtain inequality (2). However, the yield stress σ_{ji} on the yield surface is now linked with the initial stress σ_{ji}^* by a special stress-path along which p is kept fixed. Hence, *the choice of σ_{ji}^* is not arbitrary*, and (i) and (ii) do not follow this time. Let us write the deviators of σ_{ji} and \dot{e}_{ji}^p , respectively, as

$$\bar{\sigma}_{ji} = \sigma_{ji} - \frac{1}{3} \delta_{ji} \sigma_{kk}, \quad \dot{e}_{ji}^p = \dot{e}_{ji}^p - \frac{1}{3} \delta_{ji} \dot{e}_{kk}^p, \quad (4)$$

where δ_{ji} is the Kronecker delta. Inequality (2) is rewritten as

$$(\bar{\sigma}_{ji} - \bar{\sigma}_{ji}^*) \dot{e}_{ji}^p + \frac{1}{3} (\sigma_{ii} - \sigma_{ii}^*) \dot{e}_{kk}^p \geq 0. \quad (5)$$

The second term vanishes according to our postulate. Hence, we have

$$(\bar{\sigma}_{ji} - \bar{\sigma}_{ji}^*) \dot{e}_{ji}^p \geq 0 \quad (6)$$

for an arbitrary stress deviator σ_{ji}^* that gives equilibrium stress for given fixed p . Write the yield equation in the form $f(\bar{\sigma}_{ji}, p) = 0$. Then, we can conclude the *normality condition*

$$\dot{e}_{ji}^p = \Lambda \left. \frac{\partial f}{\partial \sigma_{ji}} \right|_p \quad (7)$$

where by $\left. \frac{\partial}{\partial \sigma_{ji}} \right|_p$ is meant the differentiation with fixed p . The fact that the r.h.s. gives deviator components is readily seen from

$$\left. \frac{\partial f}{\partial \sigma_{ji}} \right|_p = \frac{\partial f}{\partial \bar{\sigma}_{jk}} \frac{\partial \bar{\sigma}_{jk}}{\partial \sigma_{ji}} = \frac{\partial f}{\partial \bar{\sigma}_{jk}} \left(\delta_{ij} \delta_{ki} - \frac{1}{3} \delta_{ji} \delta_{kk} \right). \quad (8)$$

We have now reached a new expression of the associated flow rule for materials for which only incompressible plastic deformations are admitted. Note that application of this new rule to metal plasticity does not bring about any modifications to the existing theories, because it has been usually assumed that the yield function for metals does not depend on the hydrostatic pressure. It is also clear that, with the constraining stress fixed, the microscopic rearrangement criterion of Rice [7, 8] is satisfied because the remaining stress is the conjugate force with respect to the work.

3. EQUATIONS FOR PERFECT PLASTIC FLOWS OF GRANULAR MATERIALS

In the following, we denote the strain-rate tensor by E_{ji} , i.e.

$$E_{ji} = \partial_{ij} v_n (= \dot{e}_{ji}), \quad (9)$$

where v_i is the velocity and ∂_i designates $\partial/\partial x_i$. By $()$ is indicated the symmetrization of indices. We first investigate the perfect or rigid plastic flows, so that E_{ji} is the purely plastic

strain-rate. We adopt as the yield equation the extended von Mises equation

$$f(\bar{\sigma}_{ji}, p) \equiv \sqrt{[(1/2)\bar{\sigma}_{ji}\bar{\sigma}_{ji}] - \alpha p - k} = 0, \quad (10)$$

which Drucker and Prager [2] introduced (see Appendix). This is also an outcome of Kanatani's statistical theory of particle flow [15]. Application of the associated flow rule in our sense to eqn (10) yields

$$E_{ji} = \frac{\Lambda}{2} \frac{\bar{\sigma}_{ji}}{\alpha p + k}. \quad (11)$$

The scalar quantity Λ is determined by solving eqn (11) in terms of $\bar{\sigma}_{ji}$ and substituting it in eqn (10). We then obtain $\Lambda = \pm \sqrt{2E_{ji}E_{ji}}$. The positive sign is chosen in order that the specific rate of energy dissipation $\bar{\sigma}_{ji}E_{ji}$ be positive. Thus, we obtain the following constitutive equation.

$$\bar{\sigma}_{ji} = \frac{\alpha p + k}{\sqrt{[(1/2)E_{jk}E_{jk}]} E_{ji}. \quad (12)$$

The right-hand side is a homogeneous form of degree 0 in E_{ji} , and hence there exists no one-to-one correspondence between the stress and the strain-rate. Clearly, coaxiality and material incompressibility are satisfied at the same time.

Consider plane deformations in order to express the two constants α and k in terms of the internal friction angle ϕ and the cohesion constant c . Let the $x - y$ plane be the shear plane and let the y -axis coincide with the principal axis of maximum compression in the $x - y$ plane. Then, we can write

$$E_{ji} = \begin{bmatrix} e & \\ & -e \\ & & 0 \end{bmatrix}, \quad \bar{\sigma}_{ji} = \begin{bmatrix} s & \\ & -s \\ & & 0 \end{bmatrix}, \quad (13)$$

where e and s are positive. Substitution of these expressions into eqn (12) yields

$$s = \alpha p + k. \quad (14)$$

From the diagram of the Mohr stress circle, we obtain

$$\alpha = \sin \phi, \quad k = c \cos \phi, \quad (15)$$

(Fig. 4). These expressions are different from those of Drucker and Prager [2] because of our assumption of incompressibility. If $\alpha = 0$ in eqn (12), the effects of hydrostatic pressure vanishes and eqn (12) reduces to the Levi-Mises equation for metal plasticity. If, on the other hand, the cohesion k is neglected, eqn (12) reduces to the equation of Kanatani's statistical theory of particle flow [15]. We call eqn (12) the extended Levi-Mises equation.

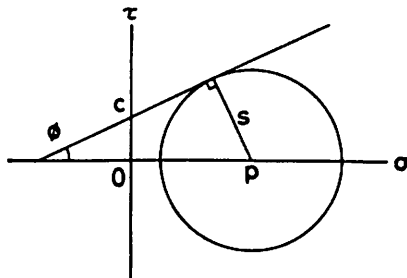


Fig. 4. The internal friction angle ϕ and the cohesion constant c .

The equation of motion for a continuum in general has the form

$$\rho \frac{dv_i}{dt} = \partial_j \sigma_{ji} + \rho b_i, \quad (16)$$

where ρ is the density and b_i the body force per unit mass. By d/dt is meant the Lagrange derivative $\partial/\partial t + v_i \partial_i$. Equation (16) into which eqn (12) is substituted and the equation of incompressibility give the following four equations for four independent variables v_x, v_y, v_z and p

$$\left. \begin{aligned} \rho \frac{dv_i}{dt} &= -\partial_i p + \frac{\alpha}{\Lambda} E_{ji} \partial_j p + \frac{\alpha}{2\Lambda} \bar{p} \Delta v_i - \frac{\alpha}{2\Lambda^3} \bar{p} E_{ji} E_{ik} \partial_j \partial_i v_k + \rho b_i, \\ \partial_i v_i &= 0. \end{aligned} \right\} \quad (17)$$

Hence, we have put $\bar{p} = p + k/\alpha$ and $\Lambda = \sqrt{[(1/2)E_{ji}E_{ji}]}$, and Δ is the Laplacian operator.

4. CHARACTERISTIC SURFACES OF PERFECT PLASTIC FLOW

Characteristic surfaces for a given set of partial differential equations are defined as follows. Suppose all the values of the derivatives appearing in the equations except those of the highest rank of differentiation for each quantity are specified on a certain surface. In general, the set of equations determines the remaining values of the highest derivatives. If they are indeterminate in particular, the surface is said to be a characteristic surface for the given data. Then, one cannot integrate the equations by specifying boundary conditions on that surface. This implies that discontinuity in the highest derivatives can arise across the surface even if all the remaining quantities are continuous across the surface. Physically, this phenomenon is understood as the *localization* of deformation, and the necessary analytical methods have been provided by Hill[19], Thomas[20], Mandel[6] *et al.* Rudnicki and Rice[21] analyzed the instability of rock deformation, considering constitutive equations that describe the dilatancy, the hardening and the vertex effect of the yield surface. Stören and Rice[22] also analyzed the localized necking in thin metal sheets by the same principle (see also [24, 5]). Here, we follow the notation of Thomas[20].

Denote the discontinuity in quantity Z by $[Z]$, i.e. $[Z] = Z^+ - Z^-$, where the superscript + refers to the side of the given unit normal and - the opposite side. Since the highest derivatives of unknown quantities in eqns (17) are $\partial_i p$ and $\partial_k \partial_j v_i$, we assume that they are discontinuous across a certain surface whose unit normal is n_i and that all the remaining quantities are continuous across the surface. This is equivalent to saying that the discontinuity is of order 1 with respect to p and of order 2 with respect to v_i . Then, we have the following geometrical compatibility conditions

$$[\partial_i p] = n_i P, \quad [\partial_k \partial_j v_i] = n_k n_j V_i. \quad (18)$$

Here, P and V_i represent the magnitude of discontinuity[19, 20]. Evaluation of discontinuity in eqns (17) then gives the following set of linear equations for P and V_i

$$\left. \begin{aligned} \left(-\frac{\Lambda^3}{\alpha} n_i + \Lambda^2 E_{ji} n_j \right) P + \frac{1}{2} (\Lambda^2 \bar{p} \delta_{ik} - \bar{p} E_{ji} E_{ik} n_j n_l) V_k &= 0, \\ n_i V_i &= 0. \end{aligned} \right\} \quad (19)$$

Non-zero discontinuity is possible only when the determinant of this set of equations vanishes. If we choose such a Cartesian coordinate system that the components of n_i become $(0, 0, 1)$, then the vanishing of the determinant of eqn (19) is

$$\begin{vmatrix} \Lambda^2 E_{xx} & \bar{p}(\Lambda^2 - E_{xx}^2)/2 & -\bar{p} E_{xx} E_{xy}/2 & -\bar{p} E_{xx} E_{xz}/2 \\ \Lambda^2 E_{xy} & -\bar{p} E_{xy} E_{xx}/2 & \bar{p}(\Lambda^2 - E_{yy}^2)/2 & -\bar{p} E_{xy} E_{yz}/2 \\ \Lambda^2 E_{xz} - \Lambda^3/\alpha & -\bar{p} E_{xz} E_{xx}/2 & -\bar{p} E_{xz} E_{xy}/2 & \bar{p}(\Lambda^2 - E_{zz}^2)/2 \\ 0 & 0 & 0 & 1 \end{vmatrix} = 0, \quad (20)$$

which reduces to

$$\Lambda^2 - \alpha E_{zz} \Lambda - (E_{xx}^2 + E_{yy}^2) = 0. \quad (21)$$

Since $n_i = (0, 0, 1)$, we can rewrite this as

$$\Lambda^2 - \alpha E_{ij} n_j \Lambda - E_{ik} E_{ik} n_j n_j + E_{ik} E_{ij} n_k n_j n_i = 0. \quad (22)$$

But this is a Cartesian tensor equation invariant to translations and rotations of the coordinate system, and hence if it holds in one particular coordinate system, it must necessarily hold in any other coordinate systems. All the terms in eqn (22) are quadratic in E_{ij} , so that they can be replaced by corresponding terms of $\bar{\sigma}_{ij}$ according to eqn (12). Taking account of eqn (10), we obtain

$$(\alpha p + k)^2 - \{\alpha(\alpha p + k) \bar{\sigma}_{ij} + \bar{\sigma}_{ik} \bar{\sigma}_{jk}\} n_j n_i + \bar{\sigma}_{ik} \bar{\sigma}_{ij} n_k n_j n_i = 0. \quad (23)$$

Considering plain deformations and putting $n_i = (\cos \theta, \sin \theta, 0)$, we substitute eqn (13) in eqn (23). Here, θ is the angle between the surface normal and the principal axis of minimum compression. Then, we obtain

$$\cos 2\theta (\cos 2\theta - \alpha) = 0. \quad (24)$$

By virtue of eqn (15), we can conclude that

$$\theta = \pm \pi/4, \pm (\pi/4 - \phi/2). \quad (25)$$

We have thus obtained two types of characteristic surfaces. One is the surfaces of maximum shearing which make the angle $\pi/4$ to the principal stress axes. The other is the stress characteristic surfaces of limit equilibrium. There has been an argument that these two types should coincide, because intuitively initial shear is thought to start along one of the stress characteristic surfaces of limit equilibrium [13, 31–36]. However, there is no definite reason why they should coincide in a developed fluid-like flow of granular materials. Moreover, our result is compatible with the experimental fact that discontinuity is observed across the surfaces of maximum shearing during plain deformations [31]. We can actually determine the amount of discontinuity. After some manipulations, we get

$$P = \pm \frac{(\alpha p + k) E_{ij} n_j n_i}{2\Lambda \sqrt{[\Lambda(\Lambda - \alpha E_{ik} n_i n_k)]}} V, \quad (26)$$

$$V_i = \mp \frac{\sqrt{[\Lambda(\Lambda - \alpha E_{ik} n_i n_k)]}}{\Lambda^2} V \left(E_{ij} n_j - \frac{\Lambda n_i - \alpha E_{ij} n_j}{\Lambda - \alpha E_{qp} n_q n_p} E_{nm} n_n n_m \right),$$

where $V = \sqrt{(V_i V_i)}$. The absolute value of V is indeterminate, because the determinant of eqns (19) to be solved is put zero. We can see that the discontinuity P vanishes on the surface of maximum shearing for plane deformation. Hence, only discontinuities with respect to the velocity field are possible across the surface of maximum shearing, which again agrees with the consequences of the statics of limit equilibrium.

5. ELASTIC-PLASTIC THEORY OF GRANULAR MATERIALS

We now extend the previous results to an elastic-plastic theory, incorporating the elastic strain as well. Let the total strain-rate E_{ij} be decomposed into the elastic part E_{ij}^e and the plastic part E_{ij}^p

$$E_{ij} = E_{ij}^e + E_{ij}^p \quad (27)$$

in such a way that the elastic strain-rate E_{ij}^e determines the stress-rate $D\sigma_{ij}/dt$. For simplicity,

we assume linearity and put

$$\frac{D\sigma_{ji}}{Dt} = 2\mu E_{ji} + \lambda \delta_{ji} E_{kk}, \quad (28)$$

where μ and λ are constants. In order that this expression be invariant to translations and rotations of the coordinate system, the time derivative $D\sigma_{ji}/Dt$ must be interpreted as

$$\frac{\partial \sigma_{ji}}{\partial t} + v_k \partial_k \sigma_{ji} - \sigma_{ki} \partial_{(k} v_{j)} - \sigma_{jk} \partial_{(k} v_{i)}, \quad (29)$$

where [] designates the alternation of indices. This time derivative is called the covariant time derivative [20] or the Jaumann–Noll derivative [26, 37]. Taking the deviator and the trace of eqn (28), we can rewrite it as

$$\frac{D\bar{\sigma}_{ji}}{Dt} = 2\mu \bar{E}_{ji}^e, \quad \frac{Dp}{Dt} = -\kappa E_{kk}^e, \quad (30)$$

where $\kappa (= (2\mu + 3\lambda)/3)$ is the bulk modulus of the material. The plastic strain-rate is, on the other hand, given by the associated flow rule in our sense, i.e.

$$\bar{E}_{ji}^p = \Lambda \left. \frac{\partial f}{\partial \sigma_{ji}} \right|_p, \quad E_{kk}^p = 0. \quad (31)$$

Combination of eqns (30) and eqns (31) yields

$$\frac{D\bar{\sigma}_{ji}}{Dt} = 2\mu \left(\bar{E}_{ji} - \Lambda \left. \frac{\partial f}{\partial \sigma_{ji}} \right|_p \right), \quad \frac{Dp}{Dt} = -\kappa E_{kk}^e. \quad (32)$$

This set of equations is an extension of the Prandtl–Reuss equation for metal plasticity, and hence we call them the extended Prandtl–Reuss equations. The scalar Λ is determined by taking the derivative Df/Dt of the yield equation $f(\bar{\sigma}_{ji}, p) = 0$. We get

$$\begin{aligned} \frac{Df}{Dt} &= \left. \frac{\partial f}{\partial \sigma_{ji}} \right|_p \frac{D\bar{\sigma}_{ji}}{Dt} + \frac{\partial f}{\partial p} \frac{Dp}{Dt} = 2\mu \left. \frac{\partial f}{\partial \sigma_{ji}} \right|_p \left(\bar{E}_{ji} - \Lambda \left. \frac{\partial f}{\partial \sigma_{ji}} \right|_p \right) - \kappa \frac{\partial f}{\partial p} E_{kk}^e = 0, \\ \therefore \Lambda &= \frac{E_{ji}^e \left. \frac{\partial f}{\partial \sigma_{ji}} \right|_p - (\kappa/2\mu) E_{kk}^e \left. \frac{\partial f}{\partial p} \right|_p}{\left. \frac{\partial f}{\partial \sigma_{ik}} \right|_p \left. \frac{\partial f}{\partial \sigma_{ik}} \right|_p}. \end{aligned} \quad (33)$$

The equation of continuity and the equations of motion are

$$\frac{d\rho}{dt} + \rho \partial_i v_i = 0, \quad \rho \frac{dv_i}{dt} = \partial_j \sigma_{ji} + \rho b_i, \quad (34)$$

respectively. Equations (32) and (34) provide ten equations for ten unknowns ρ , v_i , p and $\bar{\sigma}_{ji}$.

6. PROPAGATION OF SINGULAR SURFACES

We now investigate the propagation of singular surfaces. We assume that the derivatives appearing in eqns (32) and (34) are discontinuous across a surface which is moving in the direction of its unit normal n_i with velocity U . Hence, the singular surface is of order 1. The kinematic compatibility conditions are

$$\begin{aligned} [\partial_i \rho] &= n_i M, \quad [\partial_i v_i] = n_i V_i, \quad [\partial_k \bar{\sigma}_{ji}] = n_k \bar{\Sigma}_{ji}, \quad [\partial_k p] = n_k P, \\ [\partial \rho / \partial t] &= -UM, \quad [\partial v_i / \partial t] = -UV_i, \quad [\partial \bar{\sigma}_{ji} / \partial t] = -U\bar{\Sigma}_{ji}, \quad [\partial p / \partial t] = -UP, \end{aligned} \quad (35)$$

where M , V_i , $\bar{\Sigma}_{ji}$ and P represent the magnitude of singularity [20, 37]. Consider a particular

point on the singular surface, and take such a moving coordinate system that the material velocity at the point is zero with respect to it. Then, evaluation of discontinuities of eqns (32) and (34) at that point gives the following linear equations for M , V_i , $\bar{\Sigma}_{ji}$ and P

$$-UM + \rho n_i V_i = 0, \quad (36)$$

$$-\rho UV_i = -n_i P + n_j \bar{\Sigma}_{ji}, \quad (37)$$

$$-U \bar{\Sigma}_{ji} - \bar{\sigma}_{ki} n_{(k} V_{j)} - \bar{\sigma}_{jk} n_{(k} V_{i)} = 2\mu \left(n_{(i} V_{j)} - \frac{1}{3} \delta_{ij} n_k V_k - [\Lambda] \frac{\partial f}{\partial \sigma_{ij}} \Big|_p \right), \quad (38)$$

$$-UP = -\kappa n_k V_k, \quad (39)$$

where

$$[\Lambda] = \frac{n_j V_i \frac{\partial f}{\partial \sigma_{ij}} \Big|_p - (\kappa/2\mu) n_k V_k \frac{\partial f}{\partial p}}{\frac{\partial f}{\partial \sigma_{nm}} \Big|_p \frac{\partial f}{\partial \sigma_{nm}} \Big|_p}. \quad (40)$$

It is evident that the relation between U and n_i is obtained by putting the determinant of this system of equations to be zero. However, a little manipulation provides us a simpler form. First, eliminate P by substituting eqn (39) in eqn (37) multiplied by U . We obtain

$$U n_i \bar{\Sigma}_{ji} = -\rho U^2 V_i + \kappa n_i n_j V_j. \quad (41)$$

Eliminate $\bar{\Sigma}_{ji}$ by substituting into this equation eqn (38) multiplied by n_i . The result is written after rearrangement in the form

$$A_{ji} V_i = 0, \quad (42)$$

where

$$A_{ji} = \left(\mu - \rho U^2 + \frac{1}{2} n_i n_k \bar{\sigma}_{ik} \right) \delta_{ji} + \left(\frac{1}{3} \mu + \kappa \right) n_j n_i + n_{(i} \bar{\sigma}_{j)k} n_k - \frac{1}{2} \bar{\sigma}_{ji} - \lambda_{ji}, \quad (43)$$

$$\lambda_{ji} = \frac{2\mu n_i n_k \frac{\partial f}{\partial \sigma_{ij}} \Big|_p \frac{\partial f}{\partial \sigma_{ki}} \Big|_p - \kappa n_i n_k \frac{\partial f}{\partial \sigma_{ki}} \Big|_p \frac{\partial f}{\partial p}}{\frac{\partial f}{\partial \sigma_{nm}} \Big|_p \frac{\partial f}{\partial \sigma_{nm}} \Big|_p}. \quad (44)$$

The condition for $V_i \neq 0$ is

$$\det(A_{ji}) = 0. \quad (45)$$

This is also the necessary and sufficient condition for the existence of the singular surface when $U \neq 0$. The reason is as follows. If $V_i = 0$, then eqns (36), (38) and (39) give the trivial solution $M = 0$, $\bar{\Sigma}_{ji} = 0$ and $P = 0$. Therefore $V_i \neq 0$, and hence eqn (45) is necessary. For non-zero V_i , eqns (36), (38) and (39) determine M , $\bar{\Sigma}_{ji}$ and P . These values necessarily satisfy eqn (37), because eqn (41) is derived from eqn (39) multiplied by n_i and eqn (42). Then, eqn (37) is derived in turn from eqns (41) and (39). Thus eqn (45) is also sufficient. In the case of $U = 0$, however, there can possibly be a singular surface across which $V_i = 0$. This possibility is considered later.

If we adopt the extended von Mises eqn (10) as the yield equation, the expression for λ_{ji} becomes

$$\lambda_{ji} = \frac{\mu}{(\alpha p + k)^2} n_i n_k \bar{\sigma}_{ij} \bar{\sigma}_{ki} + \frac{\alpha \kappa}{\alpha p + k} n_j n_k \bar{\sigma}_{ki}. \quad (46)$$

Assume small shearing stress $\bar{\sigma}_{ij}/\mu \ll 1$ for simplicity, and omit those terms containing $\bar{\sigma}_{ij}$ in eqn (43). The terms in eqn (46) cannot be neglected, because they are ratios of the stress components. This approximation is equivalent to approximating the Jaumann–Noll derivatives in

eqn (32) by corresponding Lagrange derivatives [20, 21]. Consider plane motions and put $E_{xz} = E_{yz} = E_{zz} = 0$. Then, it can be seen from eqns (32) that if $\bar{\sigma}_{xx} = \bar{\sigma}_{yy} = \bar{\sigma}_{zz} = 0$ at $t = 0$, then so thereafter. Substitute eqns (13) and (14) along with $n_i = (\cos \theta, \sin \theta, 0)$ into eqn (45). We obtain

$$(\mu - \rho U^2) \left\{ (\mu - \rho U^2)^2 + \left((1 - \alpha \cos 2\theta) \kappa - \frac{2}{3} \mu \right) (\mu - \rho U^2) - \mu \left(\kappa + \frac{1}{3} \mu \right) \sin^2 2\theta \right\} = 0. \quad (47)$$

The first obvious solution is

$$U = \sqrt{(\mu/\rho)}, \quad (48)$$

which is equal to the velocity of shearing waves in the elastic domain. It is seen from eqn (42) that the discontinuity vector V_i is perpendicular to the $x - y$ plane. The remaining solutions for U are plotted in Figs. 5 and 6, where instead of α and κ we have used the internal friction angle ϕ according to eqns (15) and the Poisson ratio $\nu = \lambda/2(\lambda + \mu) = (3\kappa - 2\mu)/2(3\kappa + \mu)$. The following cases are of particular interest.

(1) $\theta = 0, \pi$. The singular surface is the surface of minimum compression. The possible velocities are

$$U = \sqrt{(\mu/\rho)}, \quad \sqrt{[(\mu/3 + (1 - \alpha)\kappa)/\rho]}. \quad (49)$$

The discontinuity vector V_i for the former velocity is tangent to the surface, whereas it is normal to the surface for the latter. Hence, the former corresponds to the shearing wave and the latter the compression wave.

(2) $\theta = \pm \pi/2$. The singular surface is the surface of maximum compression. The possible velocities are

$$U = \sqrt{(\mu/\rho)}, \quad \sqrt{[(\mu/3 + (1 + \alpha)\kappa)/\rho]}. \quad (50)$$

Again, the former corresponds to the shearing wave and the latter the compression wave.

(3) $\theta = \pm \pi/4$. The singular surface is the surface of maximum shearing stress. The possible velocities are

$$U = 0, \quad \sqrt{[(4/3)\mu + \kappa]/\rho}. \quad (51)$$

The latter is equal to the velocity of the compression wave in the elastic domain.

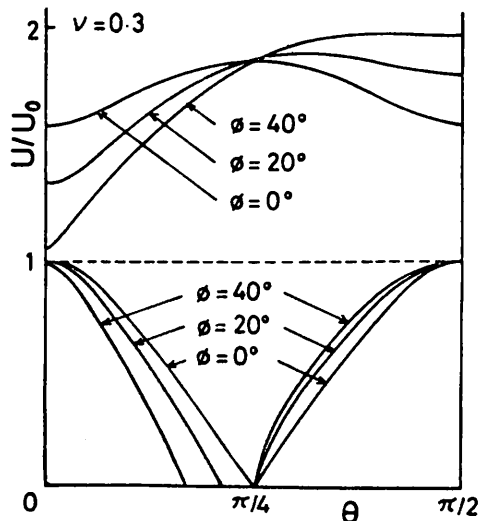


Fig. 5. The propagation speed of singular surfaces; $\nu = 0.3$, $U_0 = \sqrt{(\mu/\rho)}$.

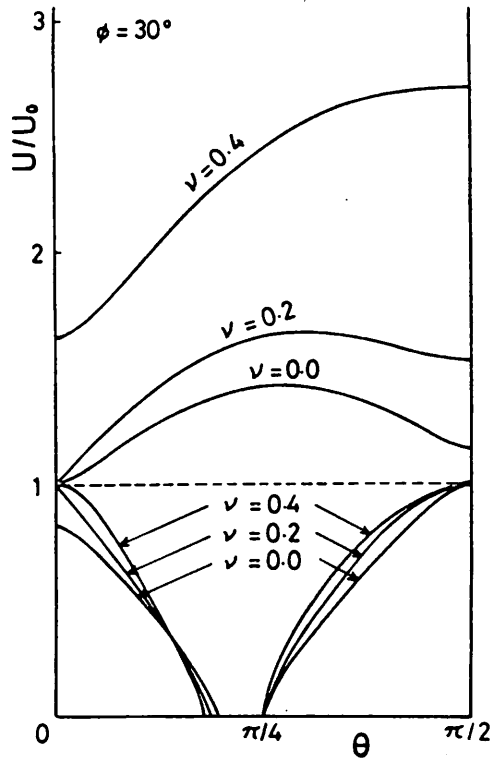


Fig. 6. The propagation speed of singular surfaces; $\phi = 30^\circ$, $U_0 = \sqrt{(\mu/\rho)}$.

(4) $U = 0$. If the singular surface is stationary, we have

$$\cos 2\theta = 0, \quad \alpha\kappa/(\kappa + \mu/3). \tag{52}$$

The first solution is $\theta = \pm \pi/4$, the surface of maximum shearing stress. The second solution $\theta = \theta^*(\phi, \nu)$ is plotted in Fig. 7. In the limit of incompressibility $\kappa/\mu \rightarrow \infty (\nu \rightarrow 0.5)$, the angle θ^* approaches $\pm(\pi/4 - \phi/2)$ as is expected. If the singular surface separates a region of plastic flow from a region in elastic limit equilibrium and is stationary, the surface must be either of the two types.

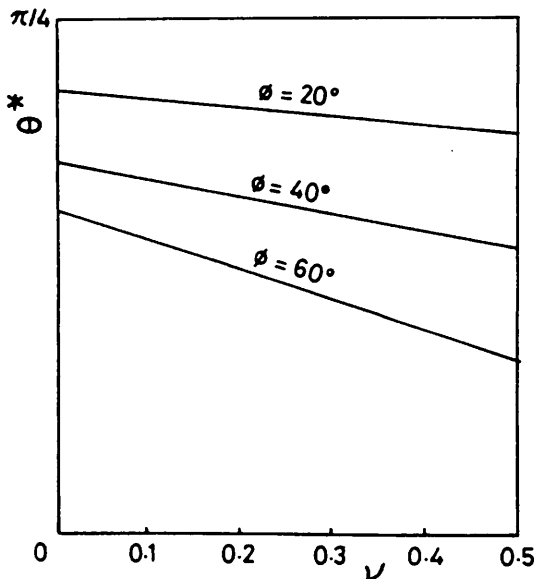


Fig. 7. The direction of stationary singular surfaces.

Finally, let us investigate the remaining possibility that $U = 0$, $V_i = 0$ and yet M , P and $\bar{\Sigma}_{ji}$ are not zero at the same time. From eqn (37)

$$n_j \bar{\Sigma}_{ji} = n_i P. \quad (53)$$

Differentiate the yield eqn (10) and evaluate discontinuity of $\partial_k f = 0$. Since the result must hold for $k = 1, 2$ and 3 , we finally obtain

$$\bar{\sigma}_{ji} \bar{\Sigma}_{ji} = 2\alpha(\alpha p + k)P. \quad (54)$$

Consider again plain deformations and put $\bar{\Sigma}_{xx} = \bar{\Sigma}_{yy} = \bar{\Sigma}_{zz} = 0$. Taking account of $\bar{\Sigma}_{yy} = -\bar{\Sigma}_{xx}$ along with $n_i = (\cos \theta, \sin \theta, 0)$, we can express eqn (53) and eqn (54) in the form

$$\begin{bmatrix} \cos \theta & \sin \theta & -\cos \theta \\ -\sin \theta & \cos \theta & -\sin \theta \\ \bar{\sigma}_{xx} - \bar{\sigma}_{yy} & 2\bar{\sigma}_{xy} & -2\alpha(\alpha p + k) \end{bmatrix} \begin{bmatrix} \bar{\Sigma}_{xx} \\ \bar{\Sigma}_{xy} \\ P \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \quad (55)$$

The vanishing of the determinant yields

$$(\bar{\sigma}_{xx} - \bar{\sigma}_{yy}) \cos 2\theta + 2\bar{\sigma}_{xy} \sin 2\theta = 2\alpha(\alpha p + k). \quad (56)$$

If we substitute (13) into this, we obtain

$$\theta = \pm (\pi/4 - \phi/2). \quad (57)$$

The singular surfaces are nothing but the stress characteristic surfaces in the limit equilibrium. We have now exhausted all the possibilities of singular surfaces of order 1.

7. CONCLUDING REMARKS

We have presented a plasticity theory for the kinematics of ideal granular materials based on a new interpretation of the associated flow rule with associated internal constraints. Adopting the extended von Mises equation as the yield equation, we have obtained the extended Levi-Mises equation in the form of a 3-dimensional tensor equation, which exhibits perfect plasticity, incompressibility and coaxiality. We have shown that the stress characteristic surface in the limit equilibrium is also the characteristic surface of the velocity equations as is intuitively expected. Besides, in plain deformations the surface of maximum shearing stress is also the characteristic surface of the velocity equations in accordance with experimental observations. Then, we have extended the theory to an elastic-plastic theory, incorporating elastic strains as well, and have obtained the extended Prandtl-Reuss equations. Finally, the singular surface propagation was analyzed and the possible velocities and directions of propagation were determined. The resulting relations hold in particular for a singular surface separating the plastic flow region from the region of elastic limit equilibrium. Thus, our theory includes the statics of limit equilibrium in it, and hence it seems to provide a proper basis for further development and applications in many problems.

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APPENDIX

Consider in the material a surface element of unit normal n_i . If the stress is σ_{ij} , the tangential stress τ_i and the normal pressure σ on the surface element are, respectively

$$\tau_i = n_j \sigma_{ji} - n_k n_l \sigma_{kl}, \quad \sigma = -n_i n_i \sigma_{ii}. \quad (\text{A1})$$

The Mohr-Coulomb yield criterion states that the material is stable as long as the Coulomb yield criterion is not met on any surface element. Hence the yield equation is expressed as

$$\max_{n_i} \{ \tau_i \tau_i - (\mu' \sigma + c')^2 \} = 0, \quad (\text{A2})$$

where μ' and c' are positive constants and n_i can assume all directions. If, on the other hand, we demand that the Coulomb criterion be satisfied on the average over all directions, we can write the yield equation as

$$\overline{\tau_i \tau_i - (\mu' \sigma + c')^2} = 0, \quad (\text{A3})$$

where the bar designates the average with respect to n_i over the whole solid angle 4π . Substitution of eqn (A1) and identities

$$\begin{aligned} \overline{n_i n_i} &= (1/3) \delta_{ii}, & \overline{n_k n_l n_i} &= 0, \\ \overline{n_i n_k n_l n_i} &= (1/15) (\delta_{ik} \delta_{il} + \delta_{il} \delta_{ik} + \delta_{ij} \delta_{kl}) \end{aligned} \quad (\text{A4})$$

in eqn (A3) yields the extended von Mises eqn (10) with

$$\alpha = \sqrt{\left(\frac{15\mu'^2}{2(3-2\mu'^2)} \right)}, \quad k = \sqrt{\left(\frac{15}{2(3-2\mu'^2)} \right)} c'. \quad (\text{A5})$$

Note that if $\mu' = 0$, then eqns (A2) and (A3) reduce, respectively, to the Tresca criterion and to the von Mises criterion for metal plasticity.