Measurement of Particle Orientation Distribution by a Stereological Method

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Abstract

Practical procedures are described for a stereological method which determines the spatial distribution of particle orientation from the distribution observed on cross sections. First, the two dimensional distribution of needle-like particles is determined by counting the number of intersections with parallel probe lines. Next, the three dimensional distribution of needle-like particles is determined by counting the number of intersections with a cutting plane. Then, the three dimensional distribution of disk-like particles is determined by counting the number of intersections with a probe line. Finally, the distribution of disk-like particles is determined by measuring the total length of the cross sections on a cutting plane.

1 Introduction

There are many natural and manufactured composite materials in which non-spherical particles, such as needle-like or disk-like ones, are distributed. In many cases, the macroscopic properties – mechanical, thermal, electrical, chemical, etc. – of such materials are greatly affected by the internal distribution of particle orientation. It is an important task, therefore, to measure the internal particle orientation distribution. However, that is not easy in practice. If needle-like particles are distributed on a plane, we can take a photograph, measure the orientation of each particle and make a histogram, but it is a tedious and painful task. Moreover, the result is sensitively affected by the length of the class interval. It is desirable, therefore, to have a systematic procedure which could also be applied by a computer system. On the other hand, if the particles are distributed within a three dimensional sample, direct counting is almost impossible. Usually, what we can observe is limited to cross-sections of a material.

Estimating three dimensional properties by observation of two dimensional cross sections is an important subject in many areas. This type of study is known as "integral geometry" or "geometrical probability" in mathematics (e. g., [1 – 3]) and as "stereology" in metallurgy, biology and medicine (e. g., [4 – 8]). In particular, the estimation of the particle size distribution of spheres from the size distribution of particle cross sections appearing on a material surface has been well studied (cf. [2 – 8]) and numerical computation schemes have also been examined in detail [9]. On the other hand, the characterization of structural anisotropy like particle orientation distribution by observation of surfaces was also studied [3 – 8] in relation to "*Buffon's* needle problem" [10]. *Hilliard* [11] first formulated it as estimation of the "distribution density". His result was further generalized in tensor formulation and described in terms of "fabric tensors" by *Kanatani* [12]. The same principle can be applied in geomechanics to estimating the crack distribution within a rock [13] and to computer image processing and artificial intelligence [14 – 16].

In this paper, we consider needle-like and disk-like particles and prescribe actual procedures for estimating the particle orientation distribution from observations of cross sections based on the general theory of *Kanatani* [12].

2 Two Dimensional Distribution of Needle-Like Particles

Suppose needle-like particles are scattered on the xy-plane. We idealize the situation by regarding each particle as a line segment with no width. Particle orientation is then specified by the angle, θ, measured from the x-axis. Since θ and $\theta + \pi$ designate the same orientation, we choose one randomly with a probability of 1/2. Then, the "distribution density" $f(\theta)$ is so defined that $f(\theta)\,d\theta$ is the total length of those particles in unit area whose orientations are between θ and $\theta + d\theta$. Hence, $c = \int_0^{2\pi} f(\theta)\,d\theta$ is the total length of the particles in unit area. If the particles are partially aligned in a certain direction, $f(\theta)$ takes on its maximum along that direction. If the particle orientation distribution is completely random, then $f(\theta)$ is constant (uniform distribution) and the entire system is macroscopically isotropic.

Although the particle orientation distribution is completely specified by the distribution density $f(\theta)$, it is not easy to determine $f(\theta)$ according to its definition as stated above. If we try to do so, we must measure the orientation of each particle and make a histogram, choosing an appropriate class interval. However, the histogram is sensitive to the choice of the class interval. If it is too large, the subsequent analysis becomes very rough. If it is too small, the result becomes unreliable. This difficulty arises from the fact that the definition of the distribution density, $f(\theta)$, involves infinitesimals or a limit taking process. On the other hand, there exists a procedure to determine $f(\theta)$ which does not involve any limit taking processes. That is the "stereological procedure" to be discussed in this paper.

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Let us put a line of orientation Θ randomly onto the plane and consider the expected number of intersections with the particles. Suppose all the particles are dissected (conceptually, of course) into infinitesimal line elements of length dl. Consider those line elements whose orientations are between θ and $\theta + d\theta$. By the definition of $f(\theta)$, there are $f(\theta)\,d\theta/dl$ such line elements in unit area. Such a line element intersects the probe line when its center falls inside the region of width $|\sin(\Theta - \theta)|d\theta$ along the line (Figure 1). Since the area of that region is $|\sin(\Theta - \theta)|d\theta$ per unit length of the probe line, there are $|\sin(\Theta - \theta)|f(\theta)d\theta$ line elements intersecting unit length of the probe line. Integrating this over all particle orientations, we find that the expected number of intersection per unit length of the probe line is given by

$$N(\Theta) = \int_0^{2\pi} |\sin(\Theta - \theta)| f(\theta)\, d\theta \,. \tag{1}$$

This was called the two dimensional "*Buffon* transform" in [13].

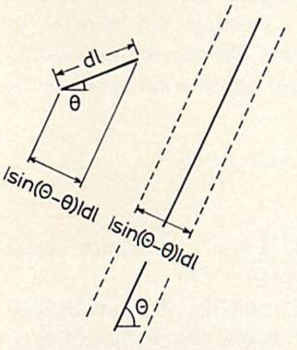

Fig. 1: A line element of length dl and orientation θ intersects a line of orientation Θ when the center of the line element falls inside the band area of width $|\sin(\Theta - \theta)|$ dl along the probe line.

3 Inverse Buffon Transform and the Fabric Tensor

The number of intersections $N(\Theta)$ between the particles and a probe line of orientation Θ can be measured for an arbitrary Θ. If we can invert the *Buffon* transform Eq. (1), we can obtain the distribution function $f(\theta)$. This is done as follows [13]. We express $N(\Theta)$ as a Fourier series:

$$N(\Theta) = \frac{C}{2\pi} = \left[1 + \sum_{n=2}^{\infty}{}' \,(A_n \cos n\Theta + B_n \sin n\Theta)\right], \tag{2}$$

$$C = \int_0^{2\pi} N(\Theta)\, d\Theta \,, \tag{3}$$

$$\begin{bmatrix} A_n \\ B_n \end{bmatrix} = \frac{2}{C}\int_0^{2\pi} N(\Theta) \begin{bmatrix} \cos n\Theta \\ \sin n\Theta \end{bmatrix} d\Theta \,. \tag{4}$$

Here, $\sum'$ denotes summation with respect to even indices. Odd terms do not appear because, by definition, $N(\Theta)$ is "symmetric" with respect to the origin, i.e., $N(\Theta) = N(\Theta + \pi)$. Then, the distribution density $f(\theta)$ is given in the following Fourier series:

$$f(\theta) = \frac{C/4}{2\pi}\left[1 - \sum_{n=2}^{\infty}{}' \,(n^2 - 1)\,(A_n \cos n\theta + B_n \sin n\theta)\right]. \tag{5}$$

(A general principle to invert Eq. (1) is discussed in [13] in terms of the rotation group, invariant operators and the group representation theory.)

It may seem that we can use the fast Fourier transform, (FFT) [19], to compute efficiently all the Fourier coefficients simultaneously, say up to a given degree N, obtaining $f(\theta)$ by Eq. (5). However, as can be seen from Eq. (5), the computation becomes very unstable if high harmonics are involved because they are greatly amplified. Moreover, in view of the statistical nature of our procedure, we are usually not interested in the details of the exact distribution. It suffices, therefore, to consider harmonics up to the second order. Hence, we are lead to the following procedure:

Take N equally spaced orientations $\Theta_k = \pi k/N$, $k = 0, 1, \ldots, N - 1$, in the interval $0 \le \Theta < \pi$. Put equally spaced parallel lines of orientation Θ_k on the plane, and let N_k be the number, per unit length, of intersections with the particles. Approximate the Fourier coefficients of Eqs. (3) and (4) by the following corresponding sums.

$$C = 2\pi \sum_{k=0}^{N-1} N_k/N \,, \tag{6}$$

$$\begin{bmatrix} A_2 \\ B_2 \end{bmatrix} = 2 \sum_{k=0}^{N-1} N_k \begin{bmatrix} \cos(2\pi k/N) \\ \sin(2\pi k/N) \end{bmatrix} \Big/ \sum_{k=0}^{N-1} N_k \,. \tag{7}$$

Then, $N(\Theta)$ is approximated by

$$N(\Theta) \simeq \frac{C}{2\pi}\left[1 + A_2 \cos 2\Theta + B_2 \sin 2\Theta\right], \tag{8}$$

and hence the distribution density $f(\theta)$ is approximated by

$$f(\theta) \simeq \frac{C/4}{2\pi}\left[1 - 3\,(A_2 \cos 2\theta + B_2 \sin 2\theta)\right]. \tag{9}$$

On the other hand, what is often required is not the form of $f(\theta)$ itself but its characteristics such as the orientation of the distribution peak, symmetries of the distribution and the discrepancy from isotropy. These characteristics become clear if we rewrite Eq. (9) in terms of xy-coordinates. If we put $x = \cos\theta$ and $y = \sin\theta$, Eq. (9) becomes

$$f(x, y) \simeq \frac{C/4}{2\pi}\left[1 + \sum_{i,j=1}^{2} D_{ij} x_i x_j\right], \tag{10}$$

$$(D_{ij}) = -3\begin{bmatrix} A_2 & B_2 \\ B_2 & -A_2 \end{bmatrix}, \tag{11}$$

where $x_1 = x$ and $x_2 = y$. The coefficient tensor D_{ij} is called the "fabric tensor" of the distribution in [12]. It vanishes if the distribution is isotropic. Hence, it describes the deviation from isotropy. Since it is a symmetric tensor, it has two mutually orthogonal principal axes. If we take these axes as the x – and the y – axis, D_{ij} becomes

$$\begin{bmatrix} \lambda & \\ & -\lambda \end{bmatrix}, \tag{12}$$

where λ and $-\lambda$ are associated principal values (eigenvalues). (Note that D_{ij} of Eq. (11) is a deviator tensor (traceless tensor), and the trace is an invariant.) In view of Eq. (10) or (9), this means that the distribution takes its maximum and minimum along the principal axes of D_{ij} and that λ is the ratio of increase or decrease of distribution from isotropy along that direction. Since the principal axes are mutually orthogonal, the distribution of Eq. (9) or (10) expresses "orthogonal anisotropy" and is symmetric with respect to each principal axis.

As an example, consider the distribution of Figure 2. Figure 3 shows the number of intersections with unit length of parallel lines of 18 different orientations (at 10° intervals) whose spacing

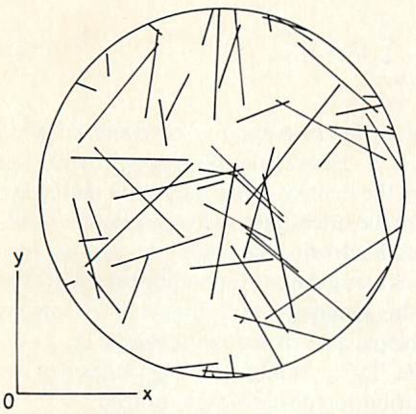

Fig. 2: A two dimensional distribution of needle-like particles.

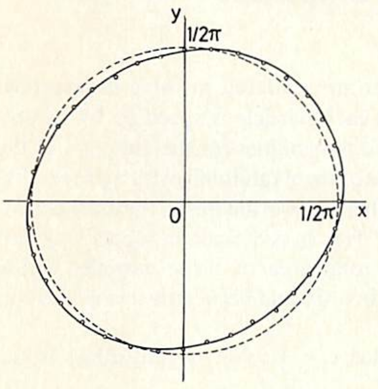

Fig. 3: Data of intersection counts on Figure 2 and the curve approximating the data distribution. Both are normalized so that the total becomes unity.

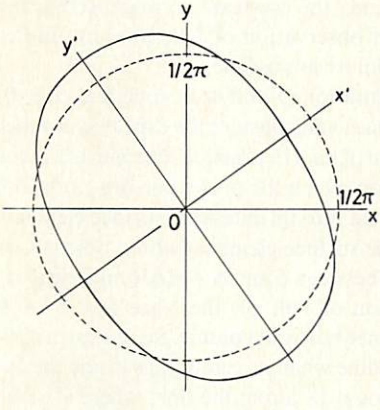

Fig. 4: The distribution density f(θ) determined from the curve of Figure 3 (normalized so that the total becomes unity).

is 1/24 the diameter of the circumference. The solid curve is the approximation of Eq. (8) and the dashed one is the circle corresponding to isotropy. However, we have normalized the data and curves so that the total becomes unity. Eq. (7) gives $A_2 = 0.026$ and $B_2 = 0.057$. Figure 4 is the corresponding shape of the (normalized) distribution density according to Eq. (9). The fabric tensor is

$$\begin{bmatrix} -0.078 & -0.171 \\ -0.171 & 0.078 \end{bmatrix} . \tag{13}$$

The orientations of the principal axes are $\theta = 35.5°$ and $125.5°$, and the associated principal values are $\lambda = \pm 1.188$, at which ratio the distribution differs from isotropy. It is difficult to guess this result by just looking at Figure 2.

4 Three Dimensional Distribution of Needle-Like Particles

Suppose needle-like particles are scattered in three dimensional space. The orientation of each particle is specified by spherical coordinates, θ, ϕ, associated with a fixed xyz-coordinate system (Figure 5). Here again, θ, ϕ and $\pi - \theta$, $\phi + \pi$ designate the same orientation, so that we choose either of them randomly with a probability of 1/2. Let the "distribution density" $f(\theta, \phi)$ be defined in such a way that $f(\theta, \phi) \sin\theta \, d\theta \, d\phi$ is the total length of those particles whose orientations lie between θ and $\theta + d\theta$ and between ϕ and $\phi + d\phi$. Hence, $c = \int_0^{2\pi} \int_0^{\pi} f(\theta, \phi) \sin\theta \, d\theta \, d\phi$ is the total length of the particles in unit volume. If the particles are partially aligned in a certain direction, $f(\theta, \phi)$ takes its maximum along that direction. If the particle orientation distribution is completely random, then $f(\theta, \phi) = \text{const.}$ (uniform distribution) and the system is macroscopically isotropic. However, direct observation of $f(\theta, \phi)$ according to its definition is almost impossible. Therefore, we consider the stereological procedure.

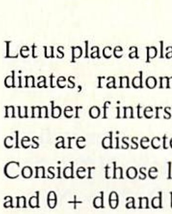

Fig. 5: Spherical coordinate system to describe the orientation of a needle-like particle.

Let us place a plane, whose unit normal is Θ, Φ in spherical coordinates, randomly into the space and consider the expected number of intersections with the particles. Imagine that all particles are dissected into infinitesimal elements of length dl. Consider those line elements whose orientations are between θ and $\theta + d\theta$ and between ϕ and $\phi + d\phi$ in spherical coordinates. By the definition of $f(\theta, \phi)$, there are $f(\theta, \phi) \sin\theta \, d\theta \, d\phi / dl$ such line elements in unit volume. Such a line element intersects the cutting plane when its center falls inside the region of width $|\cos\gamma| \, dl$ along the plane, where γ is the angle made by two unit vectors whose spherical coordinates are θ, ϕ and Θ, Φ respectively. Namely,

$$\cos\gamma = \sin\theta \sin\Theta \cos(\Phi - \phi) + \cos\theta \cos\Theta \, . \tag{14}$$

(See Figure 6.) Since the volume of that region is $|\cos\gamma| \, dl$ per unit area of the cutting plane, there are $|\cos\gamma| f(\theta, \phi) \sin\theta \, d\theta \, d\phi$ line elements intersecting unit area of the cutting plane. Integrating this over all particle orientations, we find that the expected number of intersections per unit area of the cutting plane is given by

$$N(\Theta, \Phi) = \int_0^{2\pi}\int_0^{\pi} |\cos\gamma| f(\theta, \phi) \sin\theta d\theta d\phi . \quad (15)$$

This was called the three dimensional "*Buffon* transform" in [13].

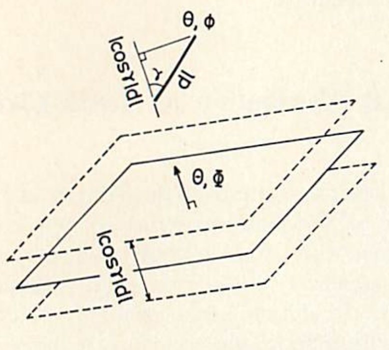

Fig. 6: A line element of length dl and orientation θ, ϕ in spherical coordinates intersects a plane whose normal is in orientation Θ, Φ in spherical coordinates when the center of the line element falls inside the layer of width |cosγ| dl along the plane.

According to [13], the inverse transform is given as follows. We express N(Θ, Φ) in the spherical harmonics series

$$N(\Theta, \Phi) = \frac{C}{4\pi}\left[1 + \sum_{n=2}^{\infty}{}'\left\{\frac{1}{2} A_{n0} P_n(\cos\Theta) + \sum_{m=1}^{n} P_n^m(\cos\Theta)\,[A_{nm}\cos m\Theta + B_{nm}\sin m\Theta]\right\}\right], \quad (16)$$

$$C = \int_0^{2\pi}\int_0^{\pi} N(\Theta, \Phi) \sin\Theta d\Theta d\Phi , \quad (17)$$

$$\begin{bmatrix} A_{nm} \\ B_{nm} \end{bmatrix} = \frac{2(2n+1)}{C}\frac{(n-m)!}{(n+m)!}\int_0^{2\pi}\int_0^{\pi} N(\Theta, \Phi) P_n^m(\cos\Theta) \begin{bmatrix} \cos m\Phi \\ \sin m\Phi \end{bmatrix} \sin\Theta d\Theta d\Phi , \quad (18)$$

where $\sum'$ again denotes summation with respect to even indices. Odd terms do not appear because, by definition, N(Θ, Φ) is "symmetric" with respect to the origin, i. e., N(Θ, Φ) = N(π − Θ, Φ + π). Here, $P_n(z)$ is the nth Legendre polynomial and $P_n^m(z)$ the associated Legendre function.

If N(Θ, Φ) is given by Eq. (16), the distribution density f(θ, ϕ) is given as follows [13]:

$$f(\theta, \phi) = \frac{c}{4\pi}\left[1 + \sum_{n=2}^{\infty}{}'\lambda_n\left\{\frac{1}{2} A_{n0} P_n(\cos\theta) + \sum_{m=1}^{n} P_n^m(\cos\theta)\,[A_{nm}\cos m\theta + B_{nm}\sin m\theta]\right\}\right], \quad (19)$$

$$c = C/2\pi , \quad (20)$$

$$\lambda_n = (-1)^{n/2-1}2^{n-1}(n-1)(n+2)/\binom{n}{n/2} . \quad (21)$$

In practice, however, it is sufficient to consider only terms up to the second order. In terms of Cartesian coordinates, Eq. (19) is rewritten in the form

$$f(x,y,z) \simeq \frac{c}{4\pi}\left[1 + \sum_{i,j=1}^{3} D_{ij}x_i x_j\right], \quad (22)$$

where x = sinθ cosϕ, y = sinθ sinϕ and z = cosθ and we put x_1 = x, x_2 = y and x_3 = z. Hence, the distribution density is completely specified by the density c and the fabric tensor D_{ij}, whose principal axes are the orientations along which the distribution takes on extremes. Each principal value is the ratio at which the distribution increases along the corresponding principal axis. The distribution has the symmetry of orthogonal anisotropy whose axes are the principal axes of the fabric tensor D_{ij}. If the distribution is isotropic, D_{ij} = 0 and hence D_{ij} measures the extent to which the distribution deviates from isotropy.

5 Three Dimensional Distribution of Disk-like Particles

Suppose disk-like particles are scattered in three dimensional space. The orientation of each particle is specified by its unit normal. Since there are two possibilities for the direction of the unit normal, we choose one of them randomly with a probability of 1/2. Let θ and ϕ be the spherical coordinates of the unit normal. The "distribution density" f(θ, ϕ) is defined in such a way that f(θ, ϕ) sinθdθdϕ is the total area of those particles whose normals lie between θ and θ + dθ and between ϕ and ϕ + dϕ in spherical coordinates. Hence, $c = \int_0^{2\pi}\int_0^{\pi} f(\theta, \phi) \sin\theta d\theta d\phi$ is the total area of the particles in unit volume. If the particle are roughly aligned, f(θ, ϕ) takes its maximum along the direction to which the particles are nearly perpendicular. If the particle orientation distribution is completely random, then f(θ, ϕ) = const. (uniform distribution) and the system is macroscopically isotropic. However, direct observation of f(θ, ϕ) according to this definition is, again, almost impossible.

Let us place a line of orientation Θ and Φ in spherical coordinates randomly into the space and consider the expected number of intersections with the particles. (In practice, one must first cut the material with a plane on which the probe line lies.) Imagine that all particles are dissected into infinitesimal surface elements of area dS. Consider those surface elements whose normals lie between θ and θ + dθ and between ϕ and ϕ + dϕ in spherical coordinates. By the definition of f(θ, ϕ), there are f(θ, ϕ) sinθ dθdϕ/dS such surface elements in unit volume. Such a surface element intersects the probe line when its center falls inside the cylindrical region of area |cosγ|dS along the line, where γ is the angle made by two unit vectors whose spherical coordinates are θ, ϕ and Θ, Φ respectively, cf. Eq. (14). (See Figure 7.) Since the volume of that region is |cosγ|dS per unit length of the probe line, there are |cosγ|f(θ, ϕ) sinθdθdϕ surface elements which intersect unit length of the probe line. Integrating this over all particle orientations, we find that the expected number of intersections per unit length of the probe line is again given by Eq. (15), i. e., the three dimensional Buffon transform, though the interpretation of f(θ, ϕ) is different. Since the basic equation is the same, subsequent procedures also proceed in the same way. Namely, if we express the observed data $N(\Theta, \Phi)$ in the spherical harmonics series of Eq. (16), the distribution density $f(\theta, \phi)$ is given by Eq. (19).

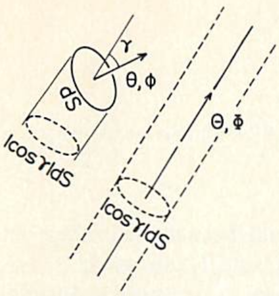

Fig. 7: A surface element of area dS whose normal has orientation θ, ϕ in spherical coordinates intersects a line of orientation Θ, Φ in spherical coordinates when the center of the surface element falls inside the cylindrical region of cross section $|\cos\gamma|\,dS$ along the probe line.

On the other hand, there is an alternative approach. Instead of a probe line, let us place a plane whose normal is Θ, Φ in spherical coordinates randomly into the space and consider the expected length of the intersections with the particles. Imagine that all particles are dissected into infinitesimal surface elements of area dS as before. Consider those surface elements whose normals lie between θ and $\theta + d\theta$ and between ϕ and $\phi + d\phi$ in spherical coordinates. Imagine that a cutting plane of unit area moves perpendicular to itself, sweeping out unit volume. Then, the average length of the intersection with a surface element as described above equals the projected area $|\sin\gamma|\,dS$ of that surface element as shown in Figure 8, where γ is the angle made by two unit vectors whose spherical coordinates are θ, ϕ and Θ, Φ, cf. Eq. (14). Hence, since there are $f(\theta, \phi)\sin\theta\, d\theta\, d\phi/dS$ such surface elements in unit volume, the expected total length of the intersections with such surface elements is $|\sin\gamma|\, f(\theta, \phi)\sin\theta\, d\theta\, d\phi$ per unit area of the cutting plane. Integrating this over all particle orientations, we find that the expected length of intersections per unit area of the cutting plane is given by

$$N(\Theta, \Phi) = \int_0^{2\pi}\int_0^{\pi} |\sin\gamma|\, f(\theta, \phi)\sin\theta\, d\theta\, d\phi\,. \tag{23}$$

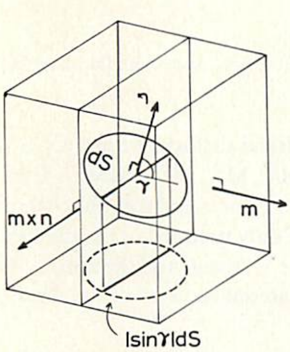

Fig. 8: If a plane of unit area whose normal has orientation Θ, Φ in spherical coordinates cuts unit volume of the space, the expected length of the intersection with a surface element whose normal has orientation θ, ϕ in spherical coordinates is equal to the area of the image projected onto a plane perpendicular to the probe plane and parallel to the intersection.

This is another type of the three dimensional Buffon transform. Inversion of Eq. (23) is similar to the previous cases [13]. Namely, if we express the observed data $N(\Theta, \Phi)$ in the spherical harmonics series of Eq. (16), the distribution density is again given by Eq. (14) except that Eqs. (20) and (21) are replaced as follows:

$$c = C/\pi^2\,, \tag{24}$$

$$\lambda_n = 2^{2n-1}(n-1)/n \binom{n}{n/2}^2\,. \tag{25}$$

Whichever approach we may take, the distribution density $f(\theta, \phi)$ is obtained in the form of Eq. (19). If higher order fluctuations are neglected, the distribution density is characterized by the density *c* and the fabric tensor D_{ij} in the form of Eq. (22). The interpretation of the fabric tensor D_{ij} is the same as before.

6 Procedures of Measurement in Practice

We have shown that the distribution density $f(\theta, \phi)$ of particle orientations, whether the particles are needle-like or disk-like, is completely determined once we know $N(\Theta, \Phi)$, i. e., the number of intersections with a probe line or a cutting plane or the length of intersections with a cutting plane. However, we can only make a finite number of observations, in other words, values of function $N(\Theta, \Phi)$ at a finite number of points on a unit sphere. Therefore, we must estimate function $N(\Theta, \Phi)$ from a finite number of values.

One procedure is the Monte Carlo method. We choose orientations randomly, choosing points on a unit sphere randomly, and observe the data for these orientations. Then, coefficients of Eq. (16) are estimated by approximating the integrals of Eqs. (17) and (18) by appropriate summations, cf. [18]. However, this means that we must cut the material with planes of various orientations. In order to do so, we must prepare a large number of material samples, all of which are supposed to have the same characteristics. This brings difficulty in many cases.

Another way is to observe only special orientations and estimate $N(\Theta, \Phi)$ for all orientations. This is impossible for a general form of $N(\Theta, \Phi)$ but is possible if the form of $N(\Theta, \Phi)$ is restricted. Suppose high order fluctuations of the distribution density $f(\theta, \phi)$ can be neglected and $f(\theta, \phi)$ is described by Eq. (22). In this case, we may say that the anisotropy is "weak". From Eqs. (16) and (19), $N(\Theta, \Phi)$ must also have the same form

$$N(\mathbf{n}) = \frac{C}{4\pi}\left[1 + \sum_{i,j=1}^{3} F_{ij}x_i x_j\right], \tag{26}$$

where we have put $\mathbf{n} = (x, y, z)$. In order to specify $N(\mathbf{n})$, we only have to know C and F_{ij}. Once they are known, we obtain the density c and the fabric tensor D_{ij} by

$$c = C/2\pi\,, \qquad D_{ij} = 4F_{ij}\,, \tag{27}$$

if the number of intersections is used and by

$$c = C/\pi^2\,, \qquad D_{ij} = -F_{ij}\,, \tag{28}$$

if the length of intersections is used.

Since there are only six unknowns (one for C and five for F_{ij}, which is a symmetric deviator (i. e., traceless) tensor), they are

determined in general if $N(\mathbf{n})$ at six different values of $\mathbf{n}$ are observed. However, this would yield unreliable results due to possible errors in the measurement. It is desirable, therefore, to compute C and F_{ij} in the form of sums or averages of a large number of observed data. This is given as follows [18].
Consider the following integrations.

$$M^{(z)} = \int_{C(z)} N(\mathbf{n})\,ds\,, \tag{29}$$

$$M_{ij}^{(z)} = \int_{C(z)} x_i x_j N(\mathbf{n})\,ds\,. \tag{30}$$

Here, C(z) is a unit circle on the xy-plane around the z-axis, and $\int_{C(s)} ds$ designates the line integral along C(z) (Figure 9). If we substitute Eq. (26), noting $F_{11} + F_{22} + F_{33} = 0$, we obtain the following expressions.

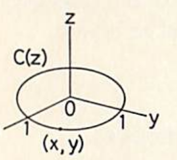

Fig. 9: Unit circle around the z-axis on the xy-plane.

$$M^{(z)} = \frac{1}{2}\left(1 - \frac{1}{2}F_{33}\right), \tag{31}$$

$$M_{12}^{(z)} = \frac{C}{8}F_{12}\,. \tag{32}$$

Define similar quantities on the yz- and the zx-plane as well. Then, we can determine C and F_{ij} in terms of them as follows.

$$C = \frac{2}{3}(M^{(x)} + M^{(y)} + M^{(z)})\,, \tag{33}$$

$$F_{11} = 2(-2M^{(x)} + M^{(y)} + M^{(z)})/(M^{(x)} + M^{(y)} + M^{(z)})\,, \tag{34}$$

$$F_{22} = 2(M^{(x)} - 2M^{(y)} + M^{(z)})/(M^{(x)} + M^{(y)} + M^{(z)})\,, \tag{35}$$

$$F_{33} = 2(M^{(x)} + M^{(y)} - 2M^{(z)})/(M^{(x)} + M^{(y)} + M^{(z)})\,, \tag{36}$$

$$F_{12} = 12M_{12}^{(z)}/(M^{(x)} + M^{(y)} + M^{(z)})(= F_{21})\,, \tag{37}$$

$$F_{23} = 12M_{23}^{(x)}/(M^{(x)} + M^{(y)} + M^{(z)}(= F_{32})\,, \tag{38}$$

$$F_{31} = 12M_{31}^{(y)}/(M^{(x)} + M^{(y)} + M^{(z)})(= F_{13})\,. \tag{39}$$

Therefore, if we know the values of quantities like Eqs. (29) and (30), we can compute C and F_{ij} by Eqs. (33) to (39). Now, we consider how to estimate quantities like Eqs. (29) and (30) from observations.
First, consider the case of disk-like particles. Cut the material randomly with a plane parallel to the xy-plane and draw on the surface a line making angle $k\pi/N$, $k = 0, 1, \ldots, N-1$ from the x-axis. Let $N_k^{(z)}$ be the number of intersections with the particles per unit length of the line (Figure 10). Then, approximate $M^{(z)}$ and $M_{ij}^{(z)}$ by

$$M_{12}^{(z)} = 2\pi \sum_{k=0}^{N-1} N_k^{(z)}/N\,, \tag{40}$$

$$M_{12}^{(z)} = \pi \sum_{k=0}^{N-1} N_k^{(z)} \sin(2\pi k/N)/N\,. \tag{41}$$

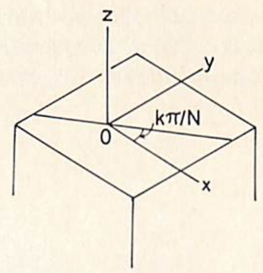

Fig. 10: A probe line drawn at angle $k\pi/N$ from the x-axis on a surface parallel to the xy-plane.

Repeat the same process for the yz- and the zx-plane and compute $M^{(x)}$, $M_{ij}^{(x)}$, $M^{(y)}$ and $M_{ij}^{(y)}$. Then, C and F_{ij} are given by Eqs. (34) to (39), and c and the fabric tensor D_{ij} are given by Eq. (27). Thus, we need only to cut the material with planes parallel to the xy-, the yz- and the zx-plane. If we use a rectangular, box-shaped, material sample, we may use its three faces alone to estimate the distribution. It is, of course, better to draw parallel lines of equal spacing instead of one line on a surface and then to cut the material with planes parallel to it to observe new surfaces successively.
Consider next the case of using cutting planes to count the number of intersections with needle-like particles or measure the length of intersections with disk-like particles. In this case, cut the material into a cylinder with the z-axis as its axis. Let $N_k^{(z)}$ be the number of intersections with needle-like particles or the length of intersections with disk-like particles observed in unit area of the strip on the cylinder whose central angle, ϕ, is in the range $2\pi(k-1/2)/N < \phi \leq 2\pi(k+1/2)/N$ (Figure 11). Then, approximate $M^{(z)}$ and $M_{ij}^{(z)}$ by

$$M^{(z)} = 2\pi \int_{k=0}^{N-1} N_k^{(z)}/N\,, \tag{42}$$

$$M_{12}^{(z)} = \pi \int_{k=0}^{N-1} N_k^{(z)} \sin(4\pi k/N)/N\,. \tag{43}$$

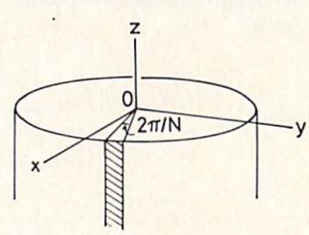

Fig. 11: A probe area of central angle $2\pi/N$ on a cylindrical surface around the z-axis.

Repeat the same process for cylindrical surfaces with the x- and the y-axis as the axes and compute $M^{(x)}$, $M_{ij}^{(x)}$, $M^{(y)}$ and $M_{ij}^{(y)}$. Then, C and F_{ij} are given by Eqs. (34) to (39), and c and the fabric tensor D_{ij} are given by Eq. (28). Thus, we only need to cut the material with cylindrical surfaces around the x-, the y- and the z-axis. We can also peel the cylinder shaped material successively to observe new surfaces.

7 Concluding Remarks

In this paper, we have shown stereological procedures to estimate the two and three dimensional orientation distributions of needle-like or disk-like particles from observations on two dimensional surfaces based on a relationship called the "Buffon transform". An important idea behind this is the concept to the "distribution density" which characterizes the particle orientation distribution. Also important is the fact that the distribution density is specified by the density c and the fabric tensor D_{ij} if higher order fluctuations are neglected. In this case, we can use a simple method requiring only three material samples. Our procedures can be performed manually but can also be implemented by a computer image processing system.

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