

A Theory of Contact Force Distribution in Granular Materials

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SUMMARY

The contact force distribution in an idealized granular material is related to the macroscopic state of the stress. The link connecting them is provided by a variational method called the principle of virtual work. The stress is expressed in terms of the contact force distribution in the material, and the contact force is then expressed in terms of the macroscopic stress. A macroscopic yield condition based on these results is also discussed. All the equations are expressed in the form of three-dimensional Cartesian tensor equations.

1. INTRODUCTION

Many types of mechanical processing of granular materials occur in industry — compression, crushing, grinding, etc. In order to analyze the performance efficiency of these operations, information on microscopic interparticle characteristics, such as the statistical distribution of contact forces, must be available. Since direct observation of such characteristics is difficult from a practical point of view, a way of estimating them from a knowledge of measurable quantities such as the macroscopic stress is necessary.

In this paper, an assembly of rigid spheres is taken as an idealized model of granular materials, and expressions describing microscopic properties in terms of macroscopic quantities are deduced. (For the mechanics of model granular materials, see, for example, refs. [1 - 3].) The particle radius is assumed to be fairly uniform and not to vary wildly from particle to particle. The link connecting them is provided by a variational method called the principle of virtual work, which demands, in general terms, that the total amount of work done by the microscopic

interparticle interactions be equal to the work described in terms of macroscopic variables. All the equations are expressed in the form of three-dimensional Cartesian tensor equations invariant to coordinate translations and rotations.

We first derive a fundamental relation between the stress and the microscopic contact force distribution. Then, we express the contact force distribution in terms of the macroscopic state of the stress. Using those expressions, we deduce a yield condition which relates local slips of the particles to overall fracture of the material.

2. STRESS AND CONTACT FORCES

Suppose the material is subject to macroscopically uniform stress. However, the interparticle forces acting on each particle may vary from particle to particle due to inhomogeneity of the material. Now, take all the contact forces on particles in some particular region in the material and superpose them on a single hypothetical sphere (Fig. 1). The radius a of the sphere is assumed to be the average radius of the particles in the region. (The precise interpretation of the average is discussed later.) We call this hypothetical sphere the representative particle, because it represents all the particles in the region in the sense of statistical average. If the number of particles in the region is sufficiently large, the distribution of the forces on the representative particle is approximated by a continuous function of the contact direction n_i . Here, n_i is the outward unit normal vector at the contact point under consideration. Let $D(n)d\Omega$ be the number of contact points contained in the differential solid angle $d\Omega$ of the representative particle divided by the

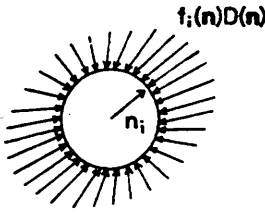


Fig. 1. Force distribution on the representative particle.

number of particles in the region under consideration. Then, $D(n)$ is the contact point density, and by definition

$$N = \int D(n) d\Omega \quad (1)$$

is the average number of contact points per single particle, which is usually referred to as the coordination number. Next, let $f_i(n)D(n)d\Omega$ be the total force acting on the differential solid angle $d\Omega$ of the representative particle divided by the number of particles in the region, and call $f_i(n)D(n)$ the contact force density. By definition, $f_i(n)$ is the average contact force per single contact point having n_i as the contact direction.

Since the particles are in mechanical equilibrium, the total vector sum of the contact forces on the representative particle should vanish:

$$\int f_i(n)D(n)d\Omega = 0 \quad (2)$$

Likewise, the torque balance demands that

$$\int f_{[i}(n)n_{j]}D(n)d\Omega = 0 \quad (3)$$

where by $[]$ we designate the alternation of tensor indices.

Now, consider the following uniform first-order deformation of the material:

$$x_i = A_{ij}x_j \quad (4)$$

according to which the point x_i moves to x'_i after deformation. For the displacement $u_i = x'_i - x_i$, we have

$$u_i = F_{ij}x_j, \quad F_{ij} \equiv A_{ij} - \delta_{ij} \quad (5)$$

where δ_{ij} is the Kronecker delta. The distortion tensor F_{ij} is resolved into the symmetric part and the skew part, *i. e.*,

$$F_{ij} = E_{ij} + R_{ij} \quad (6)$$

$$E_{ij} = F_{(ij)}, \quad R_{ij} = F_{[ij]} \quad (7)$$

where by $()$ we designate the symmetrization of tensor indices. Here, E_{ij} is the strain tensor and R_{ij} the rotation tensor. If all the particles are rigid, any deformation that does not change the microscopic state of interparticle contact is actually impossible except overall rigid rotations. However, we can hypothetically imagine this kind of deformation, *i. e.*, virtual deformations. The virtual deformation (4) distorts a 'rigid' spherical particle into an ellipsoid. Let the representative particle be subject to that virtual deformation. The displacement $\xi_i(n)$ at the contact point having n_i as the contact direction is given by

$$\xi_i(n) = aF_{ij}n_j \quad (8)$$

(Fig. 2). Since the contact forces are assumed not to change during the virtual deformation, the virtual work done by the contact forces on the representative particle is

$$\begin{aligned} \int f_i \xi_i D d\Omega &= aF_{ij} \int f_i n_j D d\Omega \\ &= aE_{ij} \int f_{(i} n_{j)} D d\Omega \end{aligned} \quad (9)$$

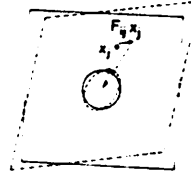


Fig. 2. Virtual deformation of granular material.

The last equality follows by virtue of the torque balance, eqn. (3). If γ is the solid volume fraction of the material, the number of particles in unit volume is $\gamma/(4/3)\pi a^3$, and this relation precisely defines the average radius a of the particles. Hence, the virtual work done in unit volume is

$$W = \frac{3}{4} \frac{\gamma}{\pi a^2} E_{ij} \int f_{(i} n_{j)} D d\Omega \quad (10)$$

Meanwhile, the virtual work done per unit volume by the virtual strain E_{ij} under the stress σ_{ij} must be

$$W = \sigma_{ij} E_{ij} \quad (11)$$

because the stress is assumed not to change during the virtual deformation. The condition that eqn. (10) and eqn. (11) always coincide is

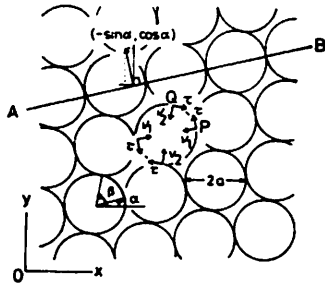


Fig. 3. Regular array of two-dimensional granular material.

$$\sigma_{ij} = \frac{3}{4} \frac{\gamma}{\pi a^2} \int f_{(i} n_{j)} D d\Omega \quad (12)$$

which gives a fundamental relation that connects the microscopic contact forces and the macroscopic stress.

The important fact to be noted is that $f_i D$ is obtained by taking averages over a certain region. If, in particular, the region is taken so as to contain only one particle, then the left-hand side of eqn. (12) is the local stress around that particle, where a is taken to be the radius of that particle, γ the local solid volume fraction and f_i the contact force on that particle. (In this case, of course, the integral is replaced by summation because the density $f_i D$ has singularities of the Dirac delta function.) This enables us to define local stress in the discrete material and to study stress fluctuations in the material. If, on the other hand, the region is taken to be the whole volume of the material, then the left-hand side of eqn. (12) is the overall average stress.

3. TWO-DIMENSIONAL EXAMPLES OF THE FUNDAMENTAL RELATION

Let us consider an idealized two-dimensional granular material, i.e. an aggregate of circular cylinders, to test that eqn. (12) represents the true relationship. In the case of two-dimensional granular materials, the number of particles in unit area is $\gamma/\pi a^2$, instead of $\gamma/(4/3)\pi a^3$, where γ is the solid area fraction and a the average radius of the circles. Hence, eqn. (12) must be modified to

$$\sigma_{ij} = \frac{\gamma}{2\pi a} \int f_{(i} n_{j)} D d\Omega \quad (13)$$

where $d\Omega$ is the plane differential angle.

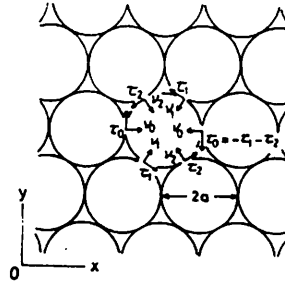


Fig. 4. Close packing of two-dimensional granular material.

Consider, as an example, the regular array of particles shown in Fig. 3. There exist two types of contact. The contact force is resolved into tangential and normal components as in Fig. 3. The two types of contact have tangential forces of the same magnitude τ due to the torque balance. The equilibrium conditions, eqns. (2) and (3), are automatically satisfied by describing the contact forces as in Fig. 3. The solid area fraction γ is clearly

$$\gamma = \pi/4 \sin(\beta - \alpha) \quad (14)$$

The integral $\int d\Omega$ in eqn. (13) is interpreted as a summation, since the force density $f_i D$ has singularities of the Dirac delta function. From eqn. (13), we finally obtain

$$\begin{aligned} \sigma_{xx} &= \frac{1}{2a} \left[\tau \cos(\beta + \alpha) - \frac{\nu_1 \cos^2 \alpha + \nu_2 \cos^2 \beta}{\sin(\beta - \alpha)} \right] \\ \sigma_{yy} &= \frac{1}{2a} \left[-\tau \cos(\beta + \alpha) + \frac{\nu_1 \sin^2 \alpha + \nu_2 \sin^2 \beta}{\sin(\beta - \alpha)} \right] \\ \sigma_{xy} &= \frac{1}{2a} \left[\tau \sin(\beta + \alpha) - \frac{\nu_1 \cos \alpha \sin \alpha + \nu_2 \cos \beta \sin \beta}{\sin(\beta - \alpha)} \right] \end{aligned} \quad (15)$$

Let us check the validity of this result by using directly the definition of stress. Consider the cross-section AB in Fig. 3. The force the upper part exerts on the lower part across the line is due to ν_2 and τ . Let F_i be the contact force at point Q in Fig. 3. Then

$$\begin{aligned} F_x &= \tau \sin \beta - \nu_2 \cos \beta \\ F_y &= -\tau \cos \beta - \nu_2 \sin \beta \end{aligned} \quad (16)$$

The contact points on the line AB are equidistant with interval $2a$ so that the force

per unit length is $(F_x/2a, F_y/2)$. The unit normal to the line is $(-\sin \alpha, \cos \alpha)$. Hence, if the macroscopic stress is σ_{ij} , we should have

$$\begin{bmatrix} F_x/2 \\ F_y/2 \end{bmatrix} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{yx} & \sigma_{yy} \end{bmatrix} \begin{bmatrix} -\sin \alpha \\ \cos \alpha \end{bmatrix} \quad (17)$$

If we substitute eqns. (15) into the right-hand side, we can confirm the validity of this relation. Hence, eqns. (15) also give the correct relation in the usual form. In the terminology of structure analysis, this example is statically determinate in the sense that we can solve eqn. (15) for τ , ν_1 and ν_2 in terms of σ_{xx} , σ_{xy} and σ_{yy} .

Consider another example shown in Fig. 4. The solid area fraction is

$$\gamma = \sqrt{3}\pi/6 \quad (18)$$

We can obtain the macroscopic stress σ_{ij} from eqn. (13) in the same way.

$$\begin{aligned} \sigma_{xx} &= \frac{1}{4a} \left[-(\tau_1 - \tau_2) - \frac{1}{\sqrt{3}}(4\nu_0 + \nu_1 + \nu_2) \right] \\ \sigma_{yy} &= \frac{1}{4} \left[(\tau_1 - \tau_2) - \sqrt{3}(\nu_1 + \nu_2) \right] \\ \sigma_{xy} &= \frac{1}{4a} \left[-\sqrt{3}(\tau_1 + \tau_2) - (\nu_1 - \nu_2) \right] \end{aligned} \quad (19)$$

This case is, however, statically indeterminate, *i.e.* we cannot uniquely determine τ_1 , τ_2 , ν_0 , ν_1 and ν_2 in terms of σ_{xx} , σ_{xy} and σ_{yy} , and hence different microscopic states of contact force can represent the same macroscopic states of the stress. In Fig. 4, the coordinate system is set in a particular manner. However, if the coordinate axes are rotated by a certain angle, the stress calculated by eqn. (13) is necessarily the same as is obtained by applying the tensor transformation rule to eqns. (19), because eqn. (13) is a Cartesian tensor equation and hence is invariant to coordinate rotations.

4. CONTACT FORCE IN TERMS OF THE STRESS

The force density $f_i D$ on the representative particle is thought of as a smooth function when the average is taken over a sufficiently large number of randomly packed particles. Hence, it is expanded into series of spherical harmonics. In our Cartesian tensor notation, the expansion has the form

$$f_i D = A_i + B_{ij} n_j + C_{ijk} n_j n_k + \dots \quad (20)$$

Retaining only the first two terms, omitting the higher harmonics, substituting this into the equilibrium conditions, eqns. (2) and (3), and using identities

$$\oint n_i d\Omega = 0, \quad \oint n_i n_j d\Omega = \frac{4}{3} \pi \delta_{ij} \quad (21)$$

we obtain

$$A_i = 0, \quad B_{\{ij\}} = 0 \quad (22)$$

Substitution of eqn. (20) into the fundamental relation, eqn. (12), then yields

$$\sigma_{ij} = \frac{\gamma}{a^2} B_{ij} \quad (23)$$

Hence,

$$f_i D = \frac{a^2}{\gamma} \sigma_{ij} n_j \quad (24)$$

This result seems quite reasonable, if we take into account the fact that eqn. (24) gives the force per unit solid angle of the representative particle. This implies that the force per unit area is $(1/\gamma)\sigma_{ij}n_j$. If the material were a complete continuum, the force density on a plane with unit normal n_i in the material would be $\sigma_{ij}n_j$, whereas in this case it is divided by γ , the solid volume fraction, due to the existence of the voidage in the material.

Now, let us turn to the density of contact points. Experiments have shown that the contact point density takes its maximum in the direction of maximum compression and its minimum in the direction of minimum compression (*e.g.* [4, 5]). Then, it is reasonable to assume the following quadratic form.

$$D = -C(\gamma)\sigma_{ij}n_in_j \quad (25)$$

Here, $C(\gamma)$ is a positive scalar dependent on the solid volume fraction γ . (The negative sign indicates that we have adopted the usual convention that the tensile stress is positive.) In view of eqn. (1), the coordination number, which also depends on γ , is given by

$$N(\gamma) = -C(\gamma)\sigma_{ij} \oint n_in_j d\Omega = -4\pi C(\gamma)p \quad (26)$$

where $p = -\sigma_{kk}/3$ is the hydrostatic pressure. Hence $C(\gamma) = -N(\gamma)/4\pi p$ and consequently

$$D = -\frac{N(\gamma)}{4\pi p} \sigma_{ij} n_i n_j \quad (27)$$

which gives the density of contact points under a given stress. The experimental function form of $N(\gamma)$ is discussed in many books and papers (e.g. [3, 6]). Combination of eqn. (27) and eqn. (24) yields

$$f_i = -\frac{4\pi a^2}{\gamma N(\gamma)} \frac{\sigma_{ij} n_j}{\sigma_{kl} n_k n_l} \quad (28)$$

which expresses the average contact force per one contact point in a prescribed contact direction n_i in terms of the solid volume fraction γ and the stress σ_{ij} .

5. MACROSCOPIC YIELD CONDITION

Let F_i be the contact force at a particular contact point with the contact direction n_i on the representative particle and let the force be resolved into tangential and normal components, τ_i and ν respectively. From eqn. (24), they are given by

$$\tau_i = \frac{a^2}{\gamma D} (\sigma_{ij} n_j - \sigma_{jk} n_i n_j n_k) \quad (29)$$

$$\nu = -\frac{a^2}{\gamma D} \sigma_{ij} n_i n_j$$

If these forces obey the Coulomb friction law, they must satisfy the inequality

$$\sqrt{\tau_i \tau_i} < \mu' \nu + c' \quad (30)$$

where μ' is the friction coefficient of the particle and c' the cohesion constant. As we increase the external load, the stress may finally become such that the inequality sign in (30) is replaced by an equality sign. Then, the contact is said to be in a *critical state*, and slip at that contact is possible. Let us, however, assume that the material does not exhibit overall fracture until the critical condition is satisfied in all directions *on the average*, in other words, until

$$\oint \left[\tau_i \tau_i - (\mu' \nu + c')^2 \right] D d\Omega = 0 \quad (31)$$

is satisfied. Identities (21) and

$$\begin{aligned} \oint n_i n_j n_k d\Omega &= 0 \\ \oint n_i n_j n_k n_l d\Omega &= \frac{4}{15} \pi (\delta_{ij} \delta_{kl} + \delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) \end{aligned} \quad (32)$$

reduce eqn. (31) to

$$\sqrt{\frac{1}{2} \tilde{\sigma}_{ij} \tilde{\sigma}_{ij}} = \sqrt{\frac{15\mu'^2}{2(3-2\mu'^2)} p} + \frac{\gamma}{a^2} \sqrt{\frac{15}{2(3-2\mu'^2)} c'} \quad (33)$$

where $\tilde{\sigma}_{ij}$ is the stress deviator:

$$\tilde{\sigma}_{ij} = \sigma_{ij} - \frac{1}{3} \delta_{ij} \sigma_{kk} \quad (34)$$

Equation (33) coincides with the so-called extended von Mises criterion. From this, we can determine the macroscopic internal angle of friction ϕ and the macroscopic cohesion constant c in such a way that eqn. (33) is reduced to the usual Coulomb law in the case of plane deformations. We finally obtain

$$\begin{aligned} \tan \phi &= \sqrt{\frac{15}{6-19\mu'^2} \mu'} \\ c &= \frac{\gamma}{a^2} \sqrt{\frac{15}{6-19\mu'^2} c'} \end{aligned} \quad (35)$$

which relate the microscopic characteristics of the particles to macroscopic yield.

If we assume, instead of eqn. (31), that the macroscopic fracture begins whenever any contact direction becomes critical, we apparently obtain the so-called Mohr-Coulomb criterion. We can also derive relations similar to eqns. (35), but the results would be much more complicated.

6. CONCLUDING REMARKS

We have investigated the relations between the microscopic state of interparticle contact and the macroscopic state of the stress. The basic principle we used is the principle of virtual work. We have expressed the stress in terms of the contact forces and the contact forces in terms of the stress. Using these relations, we also derived a macroscopic yield criterion.

The analysis of contact force distribution of granular materials has been frequently seen in many problems such as compression of sand for molds and comminution of a solid body. Usually, the coordinate axes are fixed in a particular manner, and all the vector and

tensor quantities like forces and stresses involved in the problem are treated componentwise in reference to the particular coordinate system, which often complicates the analysis. In this paper, all the equations are expressed in the form of three-dimensional tensor equations invariant to coordinate transformations. As we have seen, the results are quite satisfactory from a logical point of view and also quite reasonable from a physical point of view, so that our way of formulation seems helpful in a wide variety of practical problems.

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