IEICE TRANS. INF. & SYST., VOL. E78–D, NO. 12 DECEMBER 1995

PAPER Special Issue on Machine Vision Applications

Reliability of Fitting a Plane to Range Data

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SUMMARY Based on a simple model for the statistical error characteristics of range sensing, a numerical scheme called *renormalization* is presented for optimally fitting a planar surface to data points obtained by range sensing. The renormalization method has the advantage that not only an optimal fit is computed but also its reliability is automatically evaluated in the form of the covariance matrix. Its effectiveness is demonstrated by numerical simulation. A scheme for visualizing the reliability of computation by means of the *primary deviation pair* is also presented.

key words: *reliability of 3-D reconstruction, range sensing, plane fitting, line fitting, statistical optimization, noise estimation*

1. Introduction

For autonomous robot operations, robots must construct a 3-D model of the environment from sensor data, and range sensing and stereo vision are widely used means of such 3-D sensing [6], [7]. Although the resolution of range sensing is in general lower than stereo vision, it has the advantage that it can directly measure the distances to objects in the absence of light sources (e.g., at night) and the correspondence detection for stereo images is not necessary. For this reason, the use of range sensing is rapidly expanding to many robotics applications.

Many objects in an indoor workspace, including walls, ceilings, and floors, have planar surfaces. Hence, the problem of reconstructing a planar surface is very important for many practical robotics applications. For this purpose, a planar surface must be fitted to range data. Since the accuracy of rage data is not so high, we must apply a fitting technique that maximizes the accuracy of the solution by considering the statistical characteristics of the noise in the range data. However, *computing an optimal fit* alone is not sufficient in real applications: we must at the same time *evaluate its reliability*, because robots are unable to take appropriate actions unless the reliability of the 3-D model of the environment is quantitatively given.

Boyer et al. [1] proposed a robust sequential estimator for fitting surfaces to noisy range data that include outliers. In order to eliminate outliers, they detected edges from range data, extracted smooth regions, and fitted surfaces determined by the AIC criterion. This method is robust to noise and outliers, but the reliability of the resulting 3-D reconstruction cannot be evaluated in quantitative terms. Takeda and Latombe [9] considered a special case where the problem can reduce to a one-dimensional problem of line fitting and computed a maximum likelihood solution. However, they did not analyze the reliability of the resulting solution. Takeda et al. [8] introduced the concept of "Sensory Uncertainty Field (SUF)" for estimating the reliability of the position of the robot computed from range data when a 3-D map of the environment is given in advance. Adopting a non-statistical approach, they first approximated the noise distribution by a fixed uncertainty region and then propagated it to 3-D computation by using empirical approximations.

In this paper, we adopt a statistical approach. Assuming a simple model for the error characteristics of range sensing, we present a numerical scheme called *renormalization* [2], [4], [6] for optimally fitting a planar surface to data points obtained by range sensing. The advantage of renormalization is that not only an optimal fit is computed but also its reliability is automatically evaluated in the form of the covariance matrix. We demonstrate its effectiveness by doing numerical simulation. We also present a scheme for visualizing the reliability of the computation by means of the *primary deviation pair* [6], [7].

2. Optimal Planar Surface Fitting

Suppose we are observing a planar surface. Let $\boldsymbol{n}$ be its (outward) unit normal to it, and d its distance from the coordinate origin (Fig. 1). The equation of the surface is written in the form

$$(\boldsymbol{n}, \boldsymbol{r}) = d, \tag{1}$$

where $\boldsymbol{r}$ is a three-dimensional position vector. In this paper, we denote the inner product of vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ by $(\boldsymbol{a}, \boldsymbol{b})$. Let us call $\{\boldsymbol{n}, d\}$ the *surface parameters*. We want to fit a planar surface to data positions $\{\boldsymbol{r}_\alpha\}$, $\alpha = 1, ..., N$, measured by a range finder. Let $\{\bar{\boldsymbol{r}}_\alpha\}$ be their true positions. We write

$$\boldsymbol{r}_\alpha = \bar{\boldsymbol{r}}_\alpha + \Delta\boldsymbol{r}_\alpha, \tag{2}$$

Manuscript received April 26, 1995.
Manuscript revised July 13, 1995.

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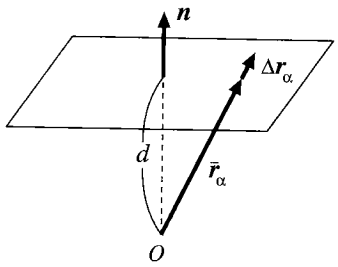


Fig. 1 Representation of a planar surface and error of range sensing.

and regard the noise $\Delta \boldsymbol{r}_\alpha$ as a Gaussian random variable of mean 0 and the covariance matrix

$$V[\boldsymbol{r}_\alpha] = E[\Delta \boldsymbol{r}_\alpha \Delta \boldsymbol{r}_\alpha^\top], \tag{3}$$

where $E[\cdot]$ denotes expectation and the superscript $\top$ denotes transpose.

Define four-dimensional vectors

$$\boldsymbol{\rho} = \begin{pmatrix} \boldsymbol{r} \\ 1 \end{pmatrix}, \quad \boldsymbol{\nu} = N\left[\begin{pmatrix} \boldsymbol{n} \\ -d \end{pmatrix}\right], \tag{4}$$

where $N[\cdot]$ denotes normalization into a unit vector. Then, the surface equation (1) is written as

$$(\boldsymbol{\nu}, \boldsymbol{\rho}) = 0. \tag{5}$$

Let $\boldsymbol{\rho}_\alpha$ be the four-dimensional representation of point $\boldsymbol{r}_\alpha$, and $\bar{\boldsymbol{\rho}}_\alpha$ its true value. The covariance matrix of $\boldsymbol{\rho}_\alpha$ is given by

$$V[\boldsymbol{\rho}_\alpha] = \begin{pmatrix} V[\boldsymbol{r}_\alpha] & \\ & 0 \end{pmatrix}. \tag{6}$$

The problem of plane fitting is stated as follows:

Problem 1: Estimate a four-dimensional unit vector $\boldsymbol{\nu}$ such that

$$(\boldsymbol{\nu}, \bar{\boldsymbol{\rho}}_\alpha) = 0, \quad \alpha = 1, ..., N, \tag{7}$$

from the data $\{\boldsymbol{\rho}_\alpha\}$, $\alpha = 1, ..., N$.

It can be shown [5] that an optimal estimator $\hat{\boldsymbol{\nu}}$ is given as the solution of the optimization

$$J[\boldsymbol{\nu}] = \sum_{\alpha=1}^{N} \frac{(\boldsymbol{\nu}, \boldsymbol{\rho}_\alpha)^2}{(\boldsymbol{\nu}, V[\boldsymbol{\rho}_\alpha]\boldsymbol{\nu})} \to \min \tag{8}$$

under the constraint $\|\boldsymbol{\nu}\| = 1$. Let $\bar{\boldsymbol{\nu}}$ be the true value of $\boldsymbol{\nu}$, and write

$$\hat{\boldsymbol{\nu}} = \bar{\boldsymbol{\nu}} + \Delta\hat{\boldsymbol{\nu}}. \tag{9}$$

The reliability of the estimator $\hat{\boldsymbol{\nu}}$ is measured by its covariance matrix

$$V[\hat{\boldsymbol{\nu}}] = E[\Delta\hat{\boldsymbol{\nu}} \Delta\hat{\boldsymbol{\nu}}^\top], \tag{10}$$

which gives a theoretical bound on attainable accuracy. It can be shown [5] that this bound is given in the form

$$V[\hat{\boldsymbol{\nu}}] = \left(\sum_{\alpha=1}^{N} \frac{\bar{\boldsymbol{\rho}}_\alpha \bar{\boldsymbol{\rho}}_\alpha^\top}{(\bar{\boldsymbol{\nu}}, V[\boldsymbol{\rho}_\alpha]\bar{\boldsymbol{\nu}})} \right)^-, \tag{11}$$

where the superscript "$-$" denotes the (*Moore-Penrose*) *generalized inverse*.

3. Error Model of Range Data

Range sensing means measuring the distances to objects from a fixed position without approaching them. A typical method is to emit a sound or radio wave and observe the phase shift between the emitted and reflected waves. The error behavior of range data depends on not only the mechanical characteristics of the range finder and the accuracy of the electronic data processing involved but also the shape and position of the object to be measured. Hence, it is very difficult to give a precise error model. If the error behavior is known for a particular range finder, we can introduce an optimization technique based on it, and the general theory in the preceding section can be applied. If the error behavior is not known, we must assume an appropriate model. Typical models are: (i) errors do not depend on the position of the object; (ii) errors increase as the distance to the object increases. The latter case is of theoretical interest, because the usual least-squares method is not optimal, as we will show shortly. In this paper, we adopt the linear model used by Takeda et al. [8] and Takeda and Latombe [9]: we assume that to a first approximation the error is proportional to the distance to the object. We also assume that the orientation in which the distance is measured can be controlled accurately.

According to this model, the error $\Delta \boldsymbol{r}_\alpha$ in Eq. (2) occurs in the direction of $\bar{\boldsymbol{r}}_\alpha$, and the covariance matrix of $\boldsymbol{r}_\alpha$ is modeled in the form

$$V[\boldsymbol{r}_\alpha] = \epsilon^2 \bar{\boldsymbol{r}}_\alpha \bar{\boldsymbol{r}}_\alpha^\top, \tag{12}$$

where ϵ is a constant, which we call the *noise level*. Since it is very difficult to predict the value of ϵ *a priori*, we treat the noise level ϵ as unknown in the subsequent computation.

The four-dimensional covariance matrix $V[\boldsymbol{\rho}_\alpha]$ can be decomposed into the noise level ϵ and the *normalized covariance matrix* $V_0[\boldsymbol{\rho}_\alpha]$ as follows:

$$V[\boldsymbol{\rho}_\alpha] = \epsilon^2 V_0[\boldsymbol{\rho}_\alpha], \quad V_0[\boldsymbol{\rho}_\alpha] = \begin{pmatrix} \bar{\boldsymbol{r}}_\alpha \bar{\boldsymbol{r}}_\alpha^\top & \\ & 0 \end{pmatrix}. \tag{13}$$

Since $(\boldsymbol{n}, \bar{\boldsymbol{r}}_\alpha) = d$, we see from Eqs. (4) that the optimization (8) can be written in the following form:

$$J[\boldsymbol{n}, d] = \frac{1}{\epsilon^2 d^2} \sum_{\alpha=1}^{N} ((\boldsymbol{n}, \boldsymbol{r}_\alpha) - d)^2 \to \min. \tag{14}$$

If the factor $1/\epsilon^2 d^2$ is ignored, this optimization reduces to the well known *least-squares method*, minimizing the sum of the squared perpendicular distances from the data positions to the surface. However, this approximation is too crude, and the resulting solution is biased. In fact, Eq. (14) implies that an optimal solution should have a larger value of d than the least-squares solution. So, we try to solve the original minimization (8) by an indirect method.

4. Removing Statistical Bias

Since multiplication of $J[\boldsymbol{\nu}]$ by a positive constant does not affect the solution of the optimization (8), the covariance matrix $V[\boldsymbol{\rho}_\alpha]$ in the denominator can be replaced by the normalized covariance $V_0[\boldsymbol{\rho}_\alpha]$. If the denominator in Eq. (8) is replaced by a constant, the optimization has the form

$$\tilde{J}[\boldsymbol{\nu}] = (\boldsymbol{\nu}, \boldsymbol{M}\boldsymbol{\nu}) \to \min, \tag{15}$$

where $\boldsymbol{M}$ is the *moment matrix* defined by

$$\boldsymbol{M} = \frac{1}{N}\sum_{\alpha=1}^{N} W_\alpha \boldsymbol{\rho}_\alpha \boldsymbol{\rho}_\alpha^\top, \tag{16}$$

$$W_\alpha = \frac{1}{(\boldsymbol{\nu}^*, V_0[\boldsymbol{\rho}_\alpha]\boldsymbol{\nu}^*)}. \tag{17}$$

Here, $\boldsymbol{\nu}^*$ is an appropriate estimate of $\boldsymbol{\nu}$. The solution of the optimization (15) is given by the unit eigenvector for the smallest eigenvalue of the moment matrix $\boldsymbol{M}$. Hence, it appears that the solution can be obtained by guessing the initial value of $\boldsymbol{\nu}$, substituting it into $\boldsymbol{\nu}^*$, updating $\boldsymbol{\nu}^*$ by the resulting solution, and iterating this process. However, the solution thus obtained is *statistically biased*. This is shown as follows.

Taking the expectation of $\boldsymbol{M}$, we obtain

$$\begin{aligned} E[\boldsymbol{M}] &= \frac{1}{N}\sum_{\alpha=1}^{N} W_\alpha E[(\bar{\boldsymbol{\rho}}_\alpha + \Delta\boldsymbol{\rho}_\alpha)(\bar{\boldsymbol{\rho}}_\alpha + \Delta\boldsymbol{\rho}_\alpha)^\top] \\ &= \bar{\boldsymbol{M}} + \frac{\epsilon^2}{N}\sum_{\alpha=1}^{N} W_\alpha V_0[\boldsymbol{\rho}_\alpha], \end{aligned} \tag{18}$$

where $\bar{\boldsymbol{M}}$ is the unperturbed moment matrix of $\boldsymbol{M}$ obtained by replacing $\boldsymbol{\rho}_\alpha$ by $\bar{\boldsymbol{\rho}}_\alpha$ in Eq. (16). We see that $\bar{\boldsymbol{M}}\boldsymbol{\nu} = \boldsymbol{0}$, i.e., $\boldsymbol{\nu}$ is the unit eigenvector of the $\bar{\boldsymbol{M}}$ for eigenvalue 0. Since $E[\boldsymbol{M}]$ is perturbed from $\bar{\boldsymbol{M}}$ by $O(\epsilon^2)$, the expectation of the computed eigenvector $\boldsymbol{\nu}$ of $\boldsymbol{M}$ is also perturbed from its true value by $O(\epsilon^2)$ according to the well known *perturbation theorem* [3]. However, if we define the *unbiased moment matrix* $\hat{\boldsymbol{M}}$ by

$$\hat{\boldsymbol{M}} = \boldsymbol{M} - \epsilon^2\boldsymbol{N}, \tag{19}$$

$$\boldsymbol{N} = \frac{1}{N}\sum_{\alpha=1}^{N} W_\alpha V_0[\boldsymbol{\rho}_\alpha], \tag{20}$$

we have $E[\hat{\boldsymbol{M}}] = \bar{\boldsymbol{M}}$. Hence, an unbiased estimator of $\boldsymbol{\nu}$ is obtained as the unit eigenvector of $\hat{\boldsymbol{M}}$ for the smallest eigenvalue.

5. Renormalization

In order to compute the unbiased moment matrix $\hat{\boldsymbol{M}}$ by Eq. (19), we need to know the noise level ϵ, but this is very difficult in practice. If we overestimate or underestimate it, the resulting solution is still biased. Also, the normalized covariance matrix $V_0[\boldsymbol{\rho}_\alpha]$ in Eq. (20) involves the true values $\bar{\boldsymbol{r}}_\alpha$, which are unknown (see Eqs. (13)). If $\bar{\boldsymbol{r}}_\alpha$ is approximated by the data value $\boldsymbol{r}_\alpha$, the error magnitude is underestimated at those points detected at shorter distances than their true positions and overestimated at those points detected at longer distances.

In order to avoid these difficulties, we apply an iterative scheme called *renormalization*, which automatically adjusts to unknown noise: the value of ϵ is estimated *a posteriori*. In the iterations, the normalized covariance matrix $V_0[\boldsymbol{\rho}_\alpha]$ is updated by approximating $\bar{\boldsymbol{r}}_\alpha$ by the projection of $\boldsymbol{r}_\alpha$ onto the fitted plane. The procedure is described as follows [2], [4]–[6]:

1. Compute the matrices

$$\boldsymbol{V}_\alpha = \begin{pmatrix} \boldsymbol{r}_\alpha \boldsymbol{r}_\alpha^\top & \\ & 0 \end{pmatrix}, \quad \alpha = 1, ..., N, \tag{21}$$

   and let $c = 0$, $V_0[\boldsymbol{\rho}_\alpha] = \boldsymbol{V}_\alpha$, and $W_\alpha = 1$, $\alpha = 1, ..., N$.

2. Compute the matrices $\boldsymbol{M}$ and $\boldsymbol{N}$ defined by Eqs. (16) and (20).

3. Compute the smallest eigenvalue λ of the matrix

$$\hat{\boldsymbol{M}} = \boldsymbol{M} - c\boldsymbol{N}, \tag{22}$$

   and the corresponding unit eigenvector $\boldsymbol{\nu}$.

4. If $\lambda \approx 0$, return $\boldsymbol{\nu} = (\nu_1, \nu_2, \nu_3, \nu_4)^\top$, c, and $\hat{\boldsymbol{M}}$. Else, update c, $V_0[\boldsymbol{\rho}_\alpha]$, and W_α as follows:

$$c \leftarrow c + \frac{\lambda}{(\boldsymbol{\nu}, \boldsymbol{N}\boldsymbol{\nu})}, \tag{23}$$

$$V_0[\boldsymbol{\rho}_\alpha] \leftarrow \frac{\nu_4^2}{((\boldsymbol{\nu}, \boldsymbol{\rho}_\alpha) - \nu_4)^2}\boldsymbol{V}_\alpha, \tag{24}$$

$$W_\alpha \leftarrow \frac{1}{(\boldsymbol{\nu}, V_0[\boldsymbol{\rho}_\alpha]\boldsymbol{\nu})}. \tag{25}$$

5. Go back to Step 2.

After renormalization, an unbiased estimator of the squared noise level ϵ^2 is obtained in the form

$$\hat{\epsilon}^2 = \frac{c}{1 - 3/N}. \tag{26}$$

Its expectation and variance are evaluated in the form

$$E[\hat{\epsilon}^2] = \epsilon^2, \quad V[\hat{\epsilon}^2] = \frac{2\epsilon^2}{N - 3}. \tag{27}$$

This is a consequence of the fact that Nc/ϵ^2 is to a first approximation a χ^2 variable with $N-3$ degrees of freedom (we omit the details).

The covariance matrix of the resulting estimator $\hat{\boldsymbol{\nu}}$ is obtained in the form

$$V[\hat{\boldsymbol{\nu}}] = \frac{\hat{\epsilon}^2}{N}(\hat{\boldsymbol{M}})_3^-, \tag{28}$$

where $(\cdot)_3^-$ denotes the generalized inverse computed after projecting the matrix onto a matrix of rank 3 by ignoring the smallest eigenvalue [5], [6]. Equation (28) is an a posteriori approximation to the theoretical bound given by Eq. (11).

6. Reliability of 3-D Reconstruction

From the computed estimate $\hat{\boldsymbol{\nu}}$, the surface parameters $\{\boldsymbol{n}, d\}$ of the plane $(\hat{\boldsymbol{\nu}}, \boldsymbol{\rho}) = 0$ are computed in the following form:

$$\hat{\boldsymbol{n}} = \frac{1}{\sqrt{1-\hat{\nu}_4{}^2}}\begin{pmatrix}\hat{\nu}_1\\ \hat{\nu}_2\\ \hat{\nu}_3\end{pmatrix}, \quad \hat{d} = -\frac{\hat{\nu}_4}{\sqrt{1-\hat{\nu}_4{}^2}}. \tag{29}$$

Once the plane is estimated, the data positions $\{\boldsymbol{r}_\alpha\}$ are *back projected* onto the estimated plane $(\hat{\boldsymbol{n}}, \boldsymbol{r}) = \hat{d}$ as follows:

$$\hat{\boldsymbol{r}}_\alpha = \frac{\hat{d}\boldsymbol{r}_\alpha}{(\hat{\boldsymbol{n}}, \boldsymbol{r}_\alpha)}. \tag{30}$$

From the covariance matrix $V[\hat{\boldsymbol{\nu}}]$ given by Eq. (28), we can compute the covariance matrix $V[\hat{\boldsymbol{n}}]$, the correlation vector $V[\hat{\boldsymbol{n}}, \hat{d}]$, and the variance $V[\hat{d}]$ as follows [5], [6]:

$$\begin{aligned} V[\hat{\boldsymbol{n}}] &= (1+\hat{d}^2)\boldsymbol{P}_{\hat{\boldsymbol{n}}} \begin{pmatrix} V[\hat{\boldsymbol{\nu}}]_{11} & V[\hat{\boldsymbol{\nu}}]_{12} & V[\hat{\boldsymbol{\nu}}]_{13} \\ V[\hat{\boldsymbol{\nu}}]_{21} & V[\hat{\boldsymbol{\nu}}]_{22} & V[\hat{\boldsymbol{\nu}}]_{23} \\ V[\hat{\boldsymbol{\nu}}]_{31} & V[\hat{\boldsymbol{\nu}}]_{32} & V[\hat{\boldsymbol{\nu}}]_{33} \end{pmatrix} \boldsymbol{P}_{\hat{\boldsymbol{n}}}, \\ V[\hat{\boldsymbol{n}}, \hat{d}] &= -(1+\hat{d}^2)^2 \boldsymbol{P}_{\hat{\boldsymbol{n}}} \begin{pmatrix} V[\hat{\boldsymbol{\nu}}]_{14} \\ V[\hat{\boldsymbol{\nu}}]_{24} \\ V[\hat{\boldsymbol{\nu}}]_{34} \end{pmatrix}, \\ V[\hat{d}] &= (1+\hat{d}^2)^3 V[\hat{\boldsymbol{\nu}}]_{44}. \end{aligned} \tag{31}$$

Here, we put $\boldsymbol{P}_{\hat{\boldsymbol{n}}} = \boldsymbol{I} - \hat{\boldsymbol{n}}\hat{\boldsymbol{n}}^\top$, which is the orthogonal projection matrix onto the plane perpendicular to $\hat{\boldsymbol{n}}$.

Since $\hat{\boldsymbol{\nu}}$ is a unit vector, errors in $\hat{\boldsymbol{\nu}}$ cannot occur in the direction $\hat{\boldsymbol{\nu}}$. This means that $V[\hat{\boldsymbol{\nu}}]$ has the following *spectral decomposition* [3]:

$$V[\hat{\boldsymbol{\nu}}] = \lambda_1 \boldsymbol{\xi}\boldsymbol{\xi}^\top + \lambda_2 \boldsymbol{\eta}\boldsymbol{\eta}^\top, \quad \lambda_1 \geq \lambda_2 > 0. \tag{32}$$

Here, λ_1 and λ_2 are the eigenvalues of $V[\hat{\boldsymbol{\nu}}]$; $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ are the corresponding unit eigenvectors. The vector $\boldsymbol{\xi}$ indicates the orientation of the most likely deviation; λ_1 is the variance in that direction. Hence, the reliability of $\hat{\boldsymbol{\nu}}$ can be visualized by displaying the two planes represented by

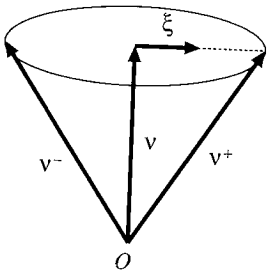


Fig. 2 Primary deviation pair.

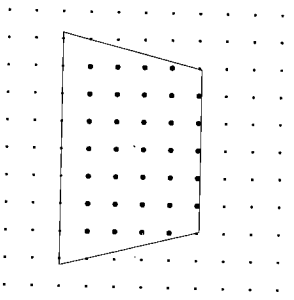


Fig. 3 Planar surface patch in the scene.

$$\boldsymbol{\nu}^+ = N[\hat{\boldsymbol{\nu}} + \sqrt{\lambda_1}\boldsymbol{\xi}], \quad \boldsymbol{\nu}^- = N[\hat{\boldsymbol{\nu}} - \sqrt{\lambda_1}\boldsymbol{\xi}]. \tag{33}$$

We call the two planes represented by $\boldsymbol{\nu}^+$ and $\boldsymbol{\nu}^-$ the *primary deviation pair* [6], [7] (Fig. 2).

7. Numerical Simulation

Figure 3 shows a planar surface patch placed in the scene and viewed from the coordinate origin, at which we assume a range finder is fixed. The dots indicate the orientations in which the distance is measured. We assume that the measurement in orientations outside the patch (indicated by small dots in Fig. 3) return the value ∞, which is ignored in the subsequent computation. We simulate the measurement process by adding Gaussian random noise to the exact distance to the surface in each orientation independently according to the statistical model given by Eq. (12), where we let $\epsilon = 0.1$.

Let $\{\bar{\boldsymbol{n}}, \bar{d}\}$ and $\{\hat{\boldsymbol{n}}, \hat{d}\}$ be the true and the computed surface parameters, respectively. Since the deviation of $\hat{\boldsymbol{n}}$ from $\bar{\boldsymbol{n}}$ is orthogonal to $\bar{\boldsymbol{n}}$ to a first approximation, the error in $\hat{\boldsymbol{n}}$ can be represented by the following vector [6]:

$$\Delta\boldsymbol{u} = \boldsymbol{P}_{\bar{\boldsymbol{n}}}(\hat{\boldsymbol{n}} - \bar{\boldsymbol{n}}) + \frac{\hat{d} - \bar{d}}{\bar{d}}\bar{\boldsymbol{n}}. \tag{34}$$

From the covariance matrix $V[\hat{\boldsymbol{n}}]$, the correlation vector $V[\hat{\boldsymbol{n}}, \hat{d}]$, and the variance $V[\hat{d}]$ computed by Eqs. (31), we can compute the covariance matrix of the vector $\Delta\boldsymbol{u}$ in the following form:

$$\begin{aligned} V[\Delta\boldsymbol{u}] = V[\hat{\boldsymbol{n}}] &+ \frac{1}{\bar{d}}(V[\hat{\boldsymbol{n}}, \hat{d}]\bar{\boldsymbol{n}}^\top + \bar{\boldsymbol{n}}V[\hat{\boldsymbol{n}}, \hat{d}]^\top) \\ &+ \frac{V[\hat{d}]}{\bar{d}^2}\bar{\boldsymbol{n}}\bar{\boldsymbol{n}}^\top. \end{aligned} \tag{35}$$

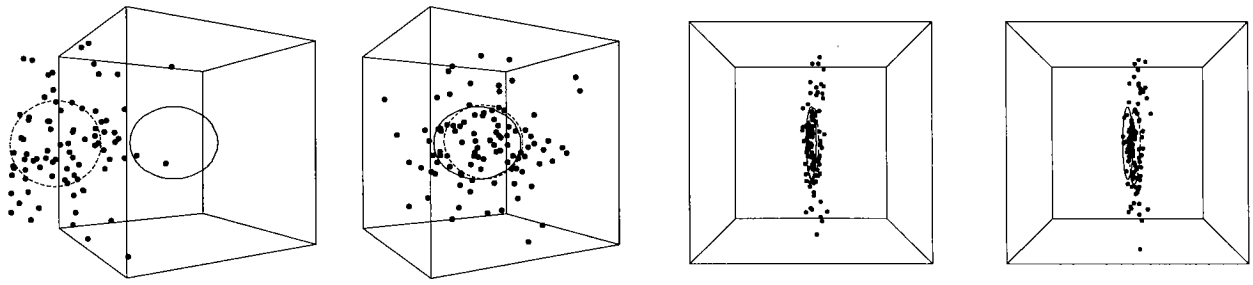


Fig. 4 Error distribution: (a) least-squares method; (b) renormalization with covariance matrix update; (c) a side view of (b); (d) renormalization without covariance matrix update (a side view).

If the distribution of $\Delta \boldsymbol{u}$ is regarded as Gaussian, the surface of equal probability in the parameter space for $\Delta \boldsymbol{u}$ is given by $(\Delta \boldsymbol{u}, V[\Delta \hat{\boldsymbol{u}}]^{-1} \Delta \boldsymbol{u}) = \text{constant}$. We call the region inside the ellipsoid

$$(\Delta \boldsymbol{u}, V[\Delta \hat{\boldsymbol{u}}]^{-1} \Delta \boldsymbol{u}) = 1 \tag{36}$$

the *standard confidence region*. In Fig. 4, the vector $\Delta \boldsymbol{u}$ is plotted in three dimensions for 100 trials, each time using different noise. Figure 4 (a) is for the least-squares method described at the end of Sect. 3. Figures 4 (b) and 4 (c) are for renormalization with covariance matrix update; Figure 4 (d) is for renormalization without covariance matrix update. The ellipses drawn in solid lines indicate the standard confidence regions computed from the theoretical expression for the covariance matrix $V[\hat{\boldsymbol{\nu}}]$ given by Eq. (11), while the ellipses drawn in dashed lines indicate the corresponding regions computed from the sample covariance matrices and centered at the sample means. We can observe that statistical bias exists in the least-squares solution and that the bias is removed by renormalization. We can also see that the covariance matrix update further increases the accuracy of the fit and the theoretical bound is almost attained.

Figure 5 (a) shows a grid pattern back projected onto a typical surface fitted by renormalization; the true position is drawn in dashed lines. Figure 5 (b) shows the primary deviation pair for the fit of Fig. 5 (a), where the a posteriori expression of Eq. (28) is used for computing the covariance matrix $V[\hat{\boldsymbol{\nu}}]$.

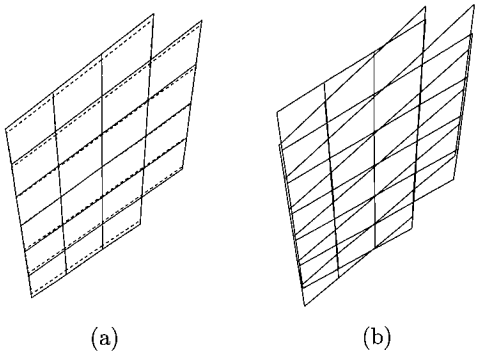


Fig. 5 (a) An example of a fit computed by renormalization. (b) Its primary deviation pair.

8. Further Applications

8.1 Hypothesis Test for Planarity

So far, we have regarded the noise level ϵ as unknown. If the accuracy of the range finder is evaluated by prior calibration, the noise level ϵ can be estimated. At the same time, its estimate $\hat{\epsilon}$ is given by Eq. (26). If the estimate $\hat{\epsilon}$ is very large as compared with the value ϵ, we can judge that the surface is not planar, i.e., the true positions of the measured points are not coplanar. This process can be formulated as a χ^2 *test* if we note the fact that Nc/ϵ^2 is approximately a χ^2 variable with $N-3$ degrees of freedom [5]. Namely, the hypothesis that the surface is planar is rejected with significance level $a\%$ if

$$\frac{\hat{\epsilon}^2}{\epsilon^2} > \frac{\chi^2_{N-3,a}}{N-3}, \tag{37}$$

where $\chi^2_{N-3,a}$ is the upper $a\%$ percent point of the χ^2 distribution with $N-3$ degrees of freedom.

8.2 Line Fitting to Range Data

As discussed by Takeda et al. [8] and Takeda and Latombe [9], the range finder is often kept horizontal and rotated in a horizontal plane. In this case, range sensing for a planar surface reduces to line fitting in the horizontal plane, and the computational procedure described above can also be applied to line fitting.

Figure 6 (a) shows an example of numerical simulation. The range finder is set at the origin O, and the measured locations are indicated by dots. The solid line is an optimal fit computed by renormalization, and the dashed line is a least-squares fit. The chained line indicates the line that passes through the true positions of the data points. Figure 6 (b) shows the corresponding primary deviation pair, indicating the reliability of

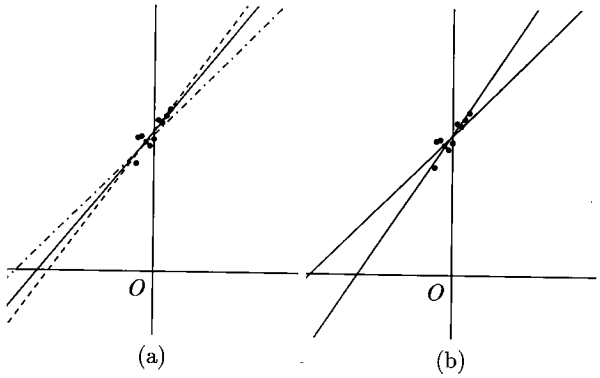


Fig. 6 (a) An example of a fit: renormalization (solid line); least-squares solution (dashed line); the true position of line (chained line). (b) Primary deviation pair.

the fit given in Fig. 6 (a).

9. Concluding Remarks

In this paper, we have assumed an idealized noise model for a range finder. As we pointed out in Sect. 3, however, range finders have different noise characteristics from device to device, so probably no real range finder exactly has the linear noise characteristics we have assumed here. The aim of this paper is (i) to point out the importance of guaranteeing optimality of 3-D reconstruction and evaluating its reliability in robotics applications, (ii) to show that for *any* noise model we can compute an optimal fit and evaluate its reliability at the same time by the renormalization technique, and (iii) to present the scheme for visualizing the reliability of 3-D reconstruction by means of the "primary deviation pair". The linear noise model was chosen as a typical example for this analysis, and the effectiveness of our approach was demonstrated by numerical simulation.

Acknowledgment

This work was in part supported by the Ministry of Education, Science, Sports and Culture, Japan under a Grant in Aid for Scientific Research B (No. 07458067) and the Okawa Institute of Information and Telecommunication.

References

[1] K.L. Boyer, M.J. Mirza, and G. Ganguly, "The robust sequential estimator: a general approach and its application to surface organization in range data," IEEE Trans. Pattern Anal. Machine Intell., vol.16, no.10, pp.987–1001, Oct. 1994.

[2] K. Kanatani, "Renormalization for unbiased estimation," Proc. 4th Int. Conf. Comput. Vision, Berlin, pp.599–606, May 1993.

[3] K. Kanatani, "Geometric Computation for Machine Vision," Oxford University Press, Oxford, U.K., 1993.

[4] K. Kanatani, "Statistical analysis of geometric computation," CVGIP: Image Understanding, vol.59, no.3, pp.286–306, May 1994.

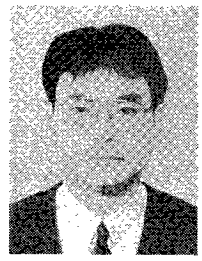
[5] K. Kanatani, "Statistical Optimization for Geometric Computation: Theory and Practice," Elsevier Science, Amsterdam, 1996.

[6] K. Kanazawa and K. Kanatani, "Direct reconstruction of planar surfaces by stereo vision," IEICE Trans. Inf. & Syst., vol.E78-D, no.7, pp.917–922, July 1995.

[7] Y. Kanazawa and K. Kanatani, "Reliability of 3-D reconstruction by stereo vision," IEICE Trans. Inf. & Syst., vol.E78-D, no.10, pp.1301–1306, Oct. 1995.

[8] H. Takeda, C. Facchinetti, and J.-C. Latombe, "Planning the motions of a mobile robot in a sensory uncertainty field," IEEE Trans. Pattern Anal. Machine Intell., vol.16, no.10, pp.1002–1017, Oct. 1994.

[9] H. Takeda and J.-C. Latombe, "Maximum likelihood fitting of a straight line to perspective range data," IEICE Trans., vol.J77-D-II, no.6, pp.1096–1103, June 1994 (in Japanese).



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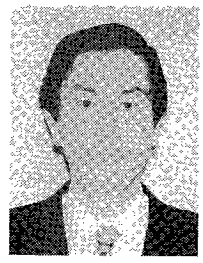

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