Unified Computation of Strict Maximum Likelihood for Geometric Fitting

Kenichi Kanatani

Department of Computer Science, Okayama University, Okayama, Japan 700-8530

ABSTRACT

A new numerical scheme is presented for strictly computing maximum likelihood (ML) of geometric fitting problems. Intensively studied in the past are those methods that first transform the data into a computationally convenient form and then assume Gaussian noise in the transformed space. In contrast, our method assumes Gaussian noise in the original data space. It is shown that the strict ML solution can be computed by iteratively using existing methods. Then, our method is applied to ellipse fitting and fundamental matrix computation. Our method is also shown to encompasses optimal correction, computing, e.g., perpendiculars to an ellipse and triangulating stereo images. While such applications have been studied individually, our method generalizes them into an application independent form from a unified point of view.

1. INTRODUCTION

This paper presents a unified numerical scheme for computing strict maximum likelihood (ML) for the problem called *geometric fitting*.[11] By "strict", we mean Gaussian noise is assumed in the original data space while it has often been assumed in the conveniently transformed data space.[13,15,17] By "unified", we mean problem-dependent schemes specifically derived in particular applications can be obtained as special cases of our general theory. To demonstrate this, we show that our theory reduces to the strict ML ellipse fitting technique of Kanatani and Sugaya[18,20] and the strict ML fundamental matrix computation of Kanatani and Sugaya.[19,21] We also show that our general theory encompasses the problem called *optimal correction*.[11] In the past, specific algorithms have been proposed for particular problems including triangulation from two views.[22] We show that these are also derived as special cases of our general theory.

We first summarize existing formations in Sect. 2, 3, and 4. Then, we present our new formulation in Sect. 5, 6, and 7, and show its applications in Sect. 8. In Sect. 9, our method is reduced to optimal correction schemes. We conclude in Sect. 10.

2. GEOMETRIC FITTING

Geometric fitting[*11] is a problem of fitting to noisy vector data $\boldsymbol{x}_\alpha$, $\alpha = 1, ..., N$, an *implicit equation* in the form

$$F(\boldsymbol{x}; \boldsymbol{u}) = 0, \tag{1}$$

parameterized by $\boldsymbol{u}$. Namely, we want to estimate the parameter $\boldsymbol{u}$ in such a way that $F(\boldsymbol{x}_\alpha; \boldsymbol{u}) \approx 0$ for all α. Many computer vision problems are formulated in this way;[10,11] one can infer the shapes and the positions of objects seen in images from the thus computed $\boldsymbol{u}$.

The function $F(\boldsymbol{x}; \boldsymbol{u})$ in Eq. (1) is generally nonlinear in the data vector $\boldsymbol{x}$. However, it is often linear in the parameter $\boldsymbol{u}$ or can be made linear by an appropriate reparameterization. In such a case, Eq. (1) can be rewritten as

$$(\boldsymbol{\xi}(\boldsymbol{x}), \boldsymbol{u}) = 0, \tag{2}$$

where and throughout this paper we denote by $(\boldsymbol{a}, \boldsymbol{b})$ the inner product of vectors $\boldsymbol{a}$ and $\boldsymbol{b}$. The ith component $\xi_i(\boldsymbol{x})$ of the vector $\boldsymbol{\xi}(\boldsymbol{x})$ consists of (generally nonlinear) terms in $\boldsymbol{x}$ that are multiplied by u_i. If terms that

E-mail: kanatani@suri.cs.okayama-u.ac.jp

[*]In statistics, it is also called the *Gauss-Helmert model* as opposed to the *Gauss-Markoff model* for which Eq. (1) can be explicitly solved for $\boldsymbol{x}$ in terms of $\boldsymbol{u}$.[6,24]

Three-Dimensional Imaging Metrology, edited by J. Angelo Beraldin, Geraldine S. Cheok, Michael McCarthy, Ulrich Neuschaefer-Rube, Proc. of SPIE-IS&T Electronic Imaging, SPIE Vol. 7239, 72390Q

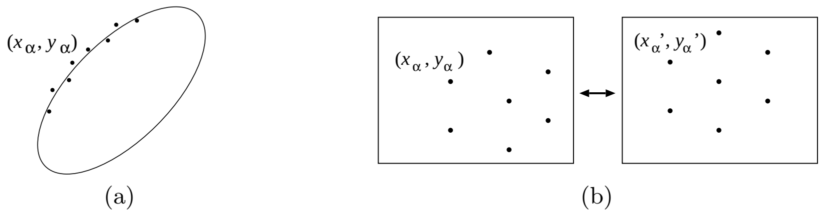


Figure 1. (a) Fitting an ellipse to a point sequence. (b) Computing the fundamental matrix from corresponding points between two images.

do not involve $\boldsymbol{u}$ are added, they are regarded as multiplied by an unknown, which we identify with the final component u_n of $\boldsymbol{u}$. Then, we should obtain a solution such that $u_n = 1$, but because Eq. (2) is homogeneous in $\boldsymbol{u}$, we can determine $\boldsymbol{u}$ only up to scale. It follows that an arbitrary normalization can be imposed on $\boldsymbol{u}$, such as $\|\boldsymbol{u}\| = 1$.

EXAMPLE 1 (ELLIPSE FITTING). We want to fit to a point sequence (x_α, y_α), $\alpha = 1, ..., N$, an ellipse in the form

$$Ax^2 + 2Bxy + Cy^2 + 2(Dx + Ey) + F = 0. \tag{3}$$

(Fig. 1(a)). If we define $\boldsymbol{\xi}(x, y)$ and $\boldsymbol{u}$ by

$$\boldsymbol{\xi}(x, y) = (x^2 \;\; 2xy \;\; y^2 \;\; 2x \;\; 2y \;\; 1)^\top, \qquad \boldsymbol{u} = (A \;\; B \;\; C \;\; D \;\; E \;\; F)^\top, \tag{4}$$

then Eq. (3) has the form of Eq. (2).[17]

EXAMPLE 2 (FUNDAMENTAL MATRIX COMPUTATION). Consider two images of the same scene viewed from different positions. If point (x, y) in the first image corresponds to (x', y') in the second, the following *epipolar equation* is satisfied[10] (Fig. 1(b)):

$$(\begin{pmatrix} x \\ y \\ 1 \end{pmatrix}, \boldsymbol{F}\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix}) = 0. \tag{5}$$

Here, $\boldsymbol{F}$ is a matrix of rank 2, called the *fundamental matrix*, which does not depend on the scene we are looking at; it depends only on the relative positions of the two cameras and their intrinsic parameters. By computing the fundamental matrix $\boldsymbol{F}$ from point correspondences, we can reconstruct the 3-D shape of the scene and the camera positions.[14] If we define

$$\boldsymbol{\xi}(x, y, x', y') = (xx' \; xy' \; x \; yx' \; yy' \; y \; x' \; y' \; 1)^\top, \qquad \boldsymbol{u} = (F_{11} \; F_{12} \; F_{13} \; F_{21} \; F_{22} \; F_{23} \; F_{31} \; F_{32} \; F_{33})^\top, \tag{6}$$

then Eq. (5) has the form of Eq. (2).[15]

3. GAUSSIAN NOISE IN THE $\boldsymbol{\xi}$-SPACE

For statistical inference from noisy data, we need to specify two things:

- Noise model: What kind of property do we assume noise to have?
- Criterion of optimality: What kind of solution do we regard as optimal?

The standard noise model is independent Gaussian noise of mean $\mathbf{0}$, for which we have two alternatives: Gaussian noise in the original data $\boldsymbol{x}_\alpha$ and Gaussian noise in the transformed data $\boldsymbol{\xi}_\alpha = \boldsymbol{\xi}(\boldsymbol{x}_\alpha)$. The aim of this paper is to compare the effect of these two. The covariance matrix $V[\boldsymbol{x}_\alpha]$ of $\boldsymbol{x}_\alpha$ and the covariance matrix $V[\boldsymbol{\xi}_\alpha]$ of $\boldsymbol{\xi}_\alpha$ are related, up to high (fourth to be exact) order terms in the noise magnitude, by

$$V[\boldsymbol{\xi}_\alpha] = \Big(\frac{\partial \boldsymbol{\xi}}{\partial \boldsymbol{x}}\Big)_\alpha V[\boldsymbol{x}_\alpha] \Big(\frac{\partial \boldsymbol{\xi}}{\partial \boldsymbol{x}}\Big)_\alpha^\top, \tag{7}$$

where $\partial\boldsymbol{\xi}/\partial\boldsymbol{x}$ is the Jacobian matrix of the mapping $\boldsymbol{\xi}(\boldsymbol{x})$, and $(\partial\boldsymbol{\xi}/\partial\boldsymbol{x})_\alpha$ means $\boldsymbol{x} = \boldsymbol{x}_\alpha$ is substituted.

EXAMPLE 3 (ELLIPSE FITTING). If each point (x_α, y_α) has independent noise of mean 0 and standard deviation σ in its x and y coordinates, the covariance matrix $V[\boldsymbol{\xi}_\alpha]$ is written as follows:[17]

$$V[\boldsymbol{\xi}_\alpha] = 4\sigma^2 \begin{pmatrix} x_\alpha^2 & x_\alpha y_\alpha & 0 & x_\alpha & 0 & 0 \\ x_\alpha y_\alpha & x_\alpha^2 + y_\alpha^2 & x_\alpha y_\alpha & y_\alpha & x_\alpha & 0 \\ 0 & x_\alpha y_\alpha & y_\alpha^2 & 0 & y_\alpha & 0 \\ x_\alpha & y_\alpha & 0 & 1 & 0 & 0 \\ 0 & x_\alpha & y_\alpha & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{8}$$

EXAMPLE 4 (FUNDAMENTAL MATRIX COMPUTATION). If each correspondence pair (x_α, y_α) and (x'_α, y'_α) has independent noise of mean 0 and standard deviation σ in its x and y coordinates, the covariance matrix $V[\boldsymbol{\xi}_\alpha]$ is written as follows:[15]

$$V[\boldsymbol{\xi}_\alpha] = \sigma^2 \begin{pmatrix} x_\alpha^2 + x_\alpha'^2 & x'_\alpha y'_\alpha & x'_\alpha & x_\alpha y_\alpha & 0 & 0 & x_\alpha & 0 & 0 \\ x'_\alpha y'_\alpha & x_\alpha^2 + y_\alpha'^2 & y'_\alpha & 0 & x_\alpha y_\alpha & 0 & 0 & x_\alpha & 0 \\ x'_\alpha & y'_\alpha & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ x_\alpha y_\alpha & 0 & 0 & y_\alpha^2 + x_\alpha'^2 & x'_\alpha y'_\alpha & x'_\alpha & y_\alpha & 0 & 0 \\ 0 & x_\alpha y_\alpha & 0 & x'_\alpha y'_\alpha & y_\alpha^2 + y_\alpha'^2 & y'_\alpha & 0 & y_\alpha & 0 \\ 0 & 0 & 0 & x'_\alpha & y'_\alpha & 1 & 0 & 0 & 0 \\ x_\alpha & 0 & 0 & y_\alpha & 0 & 0 & 1 & 0 & 0 \\ 0 & x_\alpha & 0 & 0 & y_\alpha & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{9}$$

4. ML IN THE $\boldsymbol{\xi}$-SPACE

The standard criterion for optimality is *maximum likelihood* (*ML*): the likelihood function obtained by substituting observed data into the probability density of the noise model is maximized, or equivalently its negative logarithm is minimized. It is known that the resulting solution achieves the theoretical accuracy bound called the *KCR lower bound*[3,11,12] up to higher order noise terms.

If Gaussian noise is assumed in the $\boldsymbol{\xi}$-space, ML reduces to minimization of the square sum of the *Mahalanobis distances*

$$J = \sum_{\alpha=1}^{N} (\boldsymbol{\xi}_\alpha - \bar{\boldsymbol{\xi}}_\alpha, V[\boldsymbol{\xi}_\alpha]^{-1}(\boldsymbol{\xi}_\alpha - \bar{\boldsymbol{\xi}}_\alpha)) \quad \text{subject to} \quad (\bar{\boldsymbol{\xi}}_\alpha, \boldsymbol{u}) = 0, \quad \alpha = 1, ..., N, \tag{10}$$

with respect to $\bar{\boldsymbol{\xi}}_\alpha$ and $\boldsymbol{u}$. Since the constraint is linear in $\bar{\boldsymbol{\xi}}_\alpha$, it can be eliminated by introducing Lagrange multipliers, reducing Eq. (10) to the following expression[†].[13]

$$J = \sum_{\alpha=1}^{N} \frac{(\boldsymbol{\xi}_\alpha, \boldsymbol{u})^2}{(\boldsymbol{u}, V[\boldsymbol{\xi}_\alpha]\boldsymbol{u})}. \tag{11}$$

This formulation is favored because various numerical schemes are available for minimizing Eq. (11).[13] Typical ones include the *FNS* (*Fundamental Numerical Scheme*) of Chojnacki et al.,[4] the *HEIV* (*Heteroscedastic Errors In Variables*) of Leedan and Meer,[23] and the *projective Gauss-Newton iterations* of Kanatani and Sugaya.[15,17] These apply when no special constraint (scale normalization aside) is imposed on $\boldsymbol{u}$. For computing the fundamental matrix, however, it has an additional constraint that it has rank 2. The FNS of Chojnacki et al.[4] can be extended to incorporate such constraints in the form of the *CFNS*[‡] (*Constrained FNS*) of Chojnacki et al.[5] and the *EFNS* (*Extended FNS*) of Kanatani and Sugaya.[16]

[†] If $\boldsymbol{\xi}$ has constant components as in Eqs. (4) and (6), the covariance matrix $V[\boldsymbol{\xi}_\alpha]$ becomes singular as seen in Eqs. (8) and (9). In such a case, we replace $V[\boldsymbol{\xi}_\alpha]^{-1}$ in Eq. (10) by the pseudoinverse, which means we focus only on those components of $\boldsymbol{\xi}_\alpha$ that can vary. Still, Eq. (11) holds.[11]

[‡] It was pointed out that CFNS does not necessarily compute a correct solution.[16]

5. ML IN THE x-SPACE

The preceding formulation suits numerical computation and accuracy analysis.[13] However, the assumed noise model may not always be natural. For ellipse fitting, for example, it is very natural to assume that each point (x_α, y_α) has independent Gaussian noise in its x and y coordinates. If we nonlinearly change variables as in Eqs. (4), the noise in the transformed $\boldsymbol{\xi}_\alpha$ is, strictly, no longer Gaussian. Similarly, it is natural to assume that each corresponding pair (x_α, y_α) and (x'_α, y'_α) has independent Gaussian noise in its x and y coordinates, but the noise in the nonlinearly transformed $\boldsymbol{\xi}_\alpha$ as in Eqs. (6) is, strictly, no longer Gaussian.

Whether we assume Gaussian noise in the $\boldsymbol{\xi}$-space or in the $\boldsymbol{x}$-space may not make much difference as long as the noise is small, but some difference may arise when the noise is large. Studying this is the purpose of this paper. If Gaussian noise is assumed in the $\boldsymbol{x}$-space, ML reduces to minimization of the square sum of the Mahalanobis distances[§]

$$E = \sum_{\alpha=1}^{N} (\boldsymbol{x}_\alpha - \bar{\boldsymbol{x}}_\alpha, V[\boldsymbol{x}_\alpha]^{-1}(\boldsymbol{x}_\alpha - \bar{\boldsymbol{x}}_\alpha)), \tag{12}$$

subject to the "implicit" constraint

$$(\boldsymbol{\xi}(\bar{\boldsymbol{x}}_\alpha), \boldsymbol{u}) = 0, \qquad \alpha = 1, ..., N, \tag{13}$$

with respect to $\bar{\boldsymbol{x}}_\alpha$ and $\boldsymbol{u}$. Let us call Eq. (12) the *reprojection error* (see footnote ¶).

In the past, this problem has been solved for particular applications by problem-dependent methods. For ellipse fitting, for example, auxiliary variables such as the center, the radii and the orientations of the major and minor axes, and the angles of individual points seen from the center are introduced, and the resulting high dimensional parameter space is searched by various numerical schemes.[2,7,8,25] For fundamental matrix computation, auxiliary variables are introduced by tentatively reconstructing the 3-D positions of the observed points from an assumed fundamental matrix, which is a function of the camera parameters (the relative positions of the cameras and its intrinsic parameters), and the resulting high dimensional space of these auxiliary variables is searched so that the image positions obtained by "reprojecting" the reconstructed 3-D points are as close to the observed points as possible[¶].[1] Such an approach is called *bundle adjustment*.[26]

Once we obtain an "explicit" function of unknown parameters by introducing appropriate auxiliary variables in a problem-dependent way, we can minimized it by various means such as the Levenberg-Marquardt method. However, the resulting parameter space is usually very high dimensional, so we need a clever implementation (e.g., preprocessing of sparse matrices) by considering the particularities of the problem. Recently, Kanatani and Sugaya[18,20] presented a new ML scheme for ellipse fitting in the xy plane, and Kanatani and Sugaya[19,21] presented a new ML scheme for fundamental matrix computation in the joint $xyx'y'$ space. This paper generalizes their methods and presents a problem-independent procedure for directly minimizing Eq. (12) subject to the implicit constraint of Eq. (13). No auxiliary variables are necessary.

For the problems in which the constraint has the general form of Eq. (1), Mikhail and Ackermann[6,24] described a numerical procedure based on Taylor expansion of Eq. (1) with respect to $\boldsymbol{x}$ and $\boldsymbol{u}$, which iteratively solves simultaneous linear equations in the increments of $\boldsymbol{u}$ and $\boldsymbol{x}_\alpha$, $\alpha = 1$, ..., N. Our method exploits the form of Eq. (13) and restricts the computation within the low dimensional space of the unknown $\boldsymbol{u}$.

6. FIRST APPROXIMATION

Instead of directly estimating $\bar{\boldsymbol{x}}_\alpha$, we may write

$$\bar{\boldsymbol{x}}_\alpha = \boldsymbol{x}_\alpha - \Delta\boldsymbol{x}_\alpha, \tag{14}$$

and estimate the correction term $\Delta\boldsymbol{x}_\alpha$. Then, Eq. (12) becomes

$$E = \sum_{\alpha=1}^{N} (\Delta\boldsymbol{x}_\alpha, V[\boldsymbol{x}_\alpha]^{-1}\Delta\boldsymbol{x}_\alpha). \tag{15}$$

[§]The following argument holds if $V[\boldsymbol{x}_\alpha]$ is singular. All we need is to replace $V[\boldsymbol{x}_\alpha]^{-1}$ by its pseudoinverse and appropriately use projection operations.[11]

[¶]From this, Eq. (13) comes to be known as the "reprojection error".

Equation (13) is now

$$(\boldsymbol{\xi}(\boldsymbol{x}_\alpha - \Delta\boldsymbol{x}_\alpha), \boldsymbol{u}) = 0. \tag{16}$$

Letting $\boldsymbol{\xi}_\alpha = \boldsymbol{\xi}(\boldsymbol{x}_\alpha)$, substituting the Taylor expansion $\boldsymbol{\xi}(\boldsymbol{x}_\alpha - \Delta\boldsymbol{x}_\alpha) = \boldsymbol{\xi}_\alpha - (\partial\boldsymbol{\xi}/\partial\boldsymbol{x})_\alpha \Delta\boldsymbol{x}_\alpha + \cdots$, and ignoring second order term in $\Delta\boldsymbol{x}_\alpha$, we obtain

$$(\Big(\frac{\partial\boldsymbol{\xi}}{\partial\boldsymbol{x}}\Big)_\alpha \Delta\boldsymbol{x}_\alpha, \boldsymbol{u}) = (\boldsymbol{\xi}_\alpha, \boldsymbol{u}), \tag{17}$$

where, as in Eq. (7), $(\partial\boldsymbol{\xi}/\partial\boldsymbol{x})_\alpha$ denotes the Jacobian matrix of the mapping $\boldsymbol{\xi}(\boldsymbol{x})$ followed by substitution of $\boldsymbol{x}$ $= \boldsymbol{x}_\alpha$. In order to eliminate the constraint of Eq. (17), we introduce Lagrange multipliers λ_α in the form

$$\sum_{\alpha=1}^{N} (\Delta\boldsymbol{x}_\alpha, V[\boldsymbol{x}_\alpha]^{-1}\Delta\boldsymbol{x}_\alpha) - \sum_{\alpha=1}^{N} \lambda_\alpha \Big((\Big(\frac{\partial\boldsymbol{\xi}}{\partial\boldsymbol{x}}\Big)_\alpha \Delta\boldsymbol{x}_\alpha, \boldsymbol{u}) - (\boldsymbol{\xi}_\alpha, \boldsymbol{u}) \Big). \tag{18}$$

Differentiating this with respect to $\Delta\boldsymbol{x}_\alpha$ and putting the result to $\boldsymbol{0}$, we obtain

$$2V[\boldsymbol{x}_\alpha]^{-1}\Delta\boldsymbol{x}_\alpha - \lambda_\alpha \Big(\frac{\partial\boldsymbol{\xi}}{\partial\boldsymbol{x}}\Big)_\alpha^\top \boldsymbol{u} = \boldsymbol{0}, \tag{19}$$

from which we have

$$\Delta\boldsymbol{x}_\alpha = \frac{\lambda_\alpha}{2} V[\boldsymbol{x}_\alpha] \Big(\frac{\partial\boldsymbol{\xi}}{\partial\boldsymbol{x}}\Big)_\alpha^\top \boldsymbol{u}. \tag{20}$$

Substitution of this into Eq. (17) yields

$$\frac{\lambda_\alpha}{2} (\Big(\frac{\partial\boldsymbol{\xi}}{\partial\boldsymbol{x}}\Big)_\alpha V[\boldsymbol{x}_\alpha] \Big(\frac{\partial\boldsymbol{\xi}}{\partial\boldsymbol{x}}\Big)_\alpha^\top \boldsymbol{u}, \boldsymbol{u}) = (\boldsymbol{\xi}_\alpha, \boldsymbol{u}). \tag{21}$$

If we recall the identity in Eq. (7), λ_α has the following form:

$$\lambda_\alpha = \frac{2(\boldsymbol{\xi}_\alpha, \boldsymbol{u})}{(\boldsymbol{u}, V[\boldsymbol{\xi}_\alpha]\boldsymbol{u})}. \tag{22}$$

Substituting Eq. (20) into Eq. (15), we obtain

$$E = \frac{1}{4}\sum_{\alpha=1}^{N} \lambda_\alpha^2 (V[\boldsymbol{x}_\alpha]\Big(\frac{\partial\boldsymbol{\xi}}{\partial\boldsymbol{x}}\Big)_\alpha^\top \boldsymbol{u}, \Big(\frac{\partial\boldsymbol{\xi}}{\partial\boldsymbol{x}}\Big)_\alpha^\top \boldsymbol{u}) = \frac{1}{4}\sum_{\alpha=1}^{N} \lambda_\alpha^2 (\boldsymbol{u}, \Big(\frac{\partial\boldsymbol{\xi}}{\partial\boldsymbol{x}}\Big)_\alpha V[\boldsymbol{x}_\alpha]\Big(\frac{\partial\boldsymbol{\xi}}{\partial\boldsymbol{x}}\Big)_\alpha^\top \boldsymbol{u}) = \frac{1}{4}\sum_{\alpha=1}^{N} \lambda_\alpha^2 (\boldsymbol{u}, V[\boldsymbol{\xi}_\alpha]\boldsymbol{u}). \tag{23}$$

If Eq. (22) is substituted, this becomes

$$E = \sum_{\alpha=1}^{N} \frac{(\boldsymbol{\xi}_\alpha, \boldsymbol{u})^2}{(\boldsymbol{u}, V[\boldsymbol{\xi}_\alpha]\boldsymbol{u})}, \tag{24}$$

which has the same form as Eq. (11). This means that *the first approximation of ML in the $\boldsymbol{x}$-space coincides with the strict ML in the $\boldsymbol{\xi}$-space*. Hence, Eq. (24) can be minimized by an existing method such as FNS. Let $\hat{\boldsymbol{u}}$ be the solution. From Eqs. (14), (20), and (22), the true value $\bar{\boldsymbol{x}}$ is estimated to be

$$\hat{\boldsymbol{x}}_\alpha = \boldsymbol{x}_\alpha - \frac{(\boldsymbol{\xi}_\alpha, \hat{\boldsymbol{u}})V[\boldsymbol{x}_\alpha]}{(\hat{\boldsymbol{u}}, V[\boldsymbol{\xi}_\alpha]\hat{\boldsymbol{u}})} \Big(\frac{\partial\boldsymbol{\xi}}{\partial\boldsymbol{x}}\Big)_\alpha^\top \hat{\boldsymbol{u}}. \tag{25}$$

7. HIGHER ORDER CORRECTION

The value $\hat{\boldsymbol{x}}_\alpha$ obtained by Eq. (25) is only a first approximation of the true value $\bar{\boldsymbol{x}}_\alpha$. We now put, instead of Eq. (14),

$$\bar{\boldsymbol{x}}_\alpha = \hat{\boldsymbol{x}}_\alpha - \Delta\hat{\boldsymbol{x}}_\alpha \tag{26}$$

and estimate the true value $\bar{\boldsymbol{x}}_\alpha$ by computing the correction term $\Delta\hat{\boldsymbol{x}}_\alpha$. Since $\hat{\boldsymbol{x}}_\alpha$ is a first approximation of $\bar{\boldsymbol{x}}_\alpha$, the correction term $\Delta\hat{\boldsymbol{x}}_\alpha$ is of higher order than the correction term $\Delta\boldsymbol{x}_\alpha$ in Eq. (14). Substituting Eq. (26) into Eq. (12), we can express the reprojection error E in the form

$$E = \sum_{\alpha=1}^{N} (\tilde{\boldsymbol{x}}_\alpha + \Delta\hat{\boldsymbol{x}}_\alpha, V[\boldsymbol{x}_\alpha]^{-1}(\tilde{\boldsymbol{x}}_\alpha + \Delta\hat{\boldsymbol{x}}_\alpha)), \tag{27}$$

where we put

$$\tilde{\boldsymbol{x}}_\alpha = \boldsymbol{x}_\alpha - \hat{\boldsymbol{x}}_\alpha. \tag{28}$$

Eq. (13) is written as

$$(\boldsymbol{\xi}(\hat{\boldsymbol{x}}_\alpha - \Delta\hat{\boldsymbol{x}}_\alpha), \boldsymbol{u}) = 0. \tag{29}$$

Substituting the Taylor expansion of $\boldsymbol{\xi}(\hat{\boldsymbol{x}}_\alpha - \Delta\hat{\boldsymbol{x}}_\alpha)$ and ignoring second order term in the high order quantity $\Delta\hat{\boldsymbol{x}}_\alpha$, we obtain

$$(\Big(\frac{\partial\hat{\boldsymbol{\xi}}}{\partial\boldsymbol{x}}\Big)_\alpha \Delta\hat{\boldsymbol{x}}_\alpha, \boldsymbol{u}) = (\hat{\boldsymbol{\xi}}_\alpha, \boldsymbol{u}), \tag{30}$$

where we put $\hat{\boldsymbol{\xi}}_\alpha = \boldsymbol{\xi}(\hat{\boldsymbol{x}}_\alpha)$ and $(\partial\hat{\boldsymbol{\xi}}/\partial\boldsymbol{x})_\alpha$ is the Jacobian matrix of the mapping $\boldsymbol{\xi}(\boldsymbol{x})$ followed by substitution of $\boldsymbol{x} = \hat{\boldsymbol{x}}_\alpha$. Since $\Delta\hat{\boldsymbol{x}}_\alpha$ in Eq. (26) is of higher order than $\Delta\boldsymbol{x}_\alpha$ in Eq. (14), Eq. (30) is a better approximation of Eq. (13) than Eq. (17).

In order to eliminate the constraint in Eq. (30), we introduce Lagrange multipliers λ_α in the form

$$\sum_{\alpha=1}^{N} (\tilde{\boldsymbol{x}}_\alpha + \Delta\hat{\boldsymbol{x}}_\alpha, V[\boldsymbol{x}_\alpha]^{-1}(\tilde{\boldsymbol{x}}_\alpha + \Delta\hat{\boldsymbol{x}}_\alpha)) - \sum_{\alpha=1}^{N} \lambda_\alpha \Big((\Big(\frac{\partial\hat{\boldsymbol{\xi}}}{\partial\boldsymbol{x}}\Big)_\alpha \Delta\hat{\boldsymbol{x}}_\alpha, \boldsymbol{u}) - (\hat{\boldsymbol{\xi}}_\alpha, \boldsymbol{u})\Big). \tag{31}$$

Differentiating this with respect to $\Delta\hat{\boldsymbol{x}}_\alpha$ and putting the result to $\boldsymbol{0}$, we obtain

$$2V[\boldsymbol{x}_\alpha]^{-1}(\tilde{\boldsymbol{x}}_\alpha + \Delta\hat{\boldsymbol{x}}_\alpha) - \lambda_\alpha \Big(\frac{\partial\hat{\boldsymbol{\xi}}}{\partial\boldsymbol{x}}\Big)_\alpha^\top \boldsymbol{u} = \boldsymbol{0}, \tag{32}$$

from which we have

$$\Delta\hat{\boldsymbol{x}}_\alpha = \frac{\lambda_\alpha}{2} V[\boldsymbol{x}_\alpha] \Big(\frac{\partial\hat{\boldsymbol{\xi}}}{\partial\boldsymbol{x}}\Big)_\alpha^\top \boldsymbol{u} - \tilde{\boldsymbol{x}}_\alpha. \tag{33}$$

Substitution of this into Eq. (30) yields

$$(\frac{\lambda_\alpha}{2} \Big(\frac{\partial\hat{\boldsymbol{\xi}}}{\partial\boldsymbol{x}}\Big)_\alpha V[\boldsymbol{x}_\alpha] \Big(\frac{\partial\hat{\boldsymbol{\xi}}}{\partial\boldsymbol{x}}\Big)_\alpha^\top \boldsymbol{u} - \Big(\frac{\partial\hat{\boldsymbol{\xi}}}{\partial\boldsymbol{x}}\Big)_\alpha \tilde{\boldsymbol{x}}_\alpha, \boldsymbol{u}) = (\hat{\boldsymbol{\xi}}_\alpha, \boldsymbol{u}), \tag{34}$$

which determines λ_α in the form

$$\lambda_\alpha = \frac{2(\hat{\boldsymbol{\xi}}_\alpha, \boldsymbol{u}) + 2(\boldsymbol{u}, (\partial\hat{\boldsymbol{\xi}}/\partial\boldsymbol{x})_\alpha \tilde{\boldsymbol{x}}_\alpha)}{(\boldsymbol{u}, V[\hat{\boldsymbol{\xi}}_\alpha]\boldsymbol{u})} = \frac{2(\hat{\boldsymbol{\xi}}^*_\alpha, \boldsymbol{u})}{(\boldsymbol{u}, V[\hat{\boldsymbol{\xi}}_\alpha]\boldsymbol{u})}, \tag{35}$$

where $V[\hat{\boldsymbol{\xi}}_\alpha]$ is the matrix obtained by replacing $\boldsymbol{x}_\alpha$ by $\hat{\boldsymbol{x}}_\alpha$ in $V[\boldsymbol{\xi}_\alpha]$ in Eq. (7), and we define $\hat{\boldsymbol{\xi}}^*_\alpha$ to be

$$\hat{\boldsymbol{\xi}}^*_\alpha = \hat{\boldsymbol{\xi}}_\alpha + \Big(\frac{\partial\hat{\boldsymbol{\xi}}}{\partial\boldsymbol{x}}\Big)_\alpha \tilde{\boldsymbol{x}}_\alpha. \tag{36}$$

Substituting Eq. (33) into Eq. (27), we obtain

$$E = \sum_{\alpha=1}^{N} \frac{\lambda_\alpha^2}{4} (V[\boldsymbol{x}_\alpha] \Big(\frac{\partial\hat{\boldsymbol{\xi}}}{\partial\boldsymbol{x}}\Big)_\alpha^\top \boldsymbol{u}, V[\boldsymbol{x}_\alpha]^{-1} V[\boldsymbol{x}_\alpha] \Big(\frac{\partial\hat{\boldsymbol{\xi}}}{\partial\boldsymbol{x}}\Big)_\alpha^\top \boldsymbol{u}) = \frac{1}{4} \sum_{\alpha=1}^{N} \lambda_\alpha^2 (\boldsymbol{u}, V[\hat{\boldsymbol{\xi}}_\alpha]\boldsymbol{u}). \tag{37}$$

Substituting Eq. (35) into this, we can express the reprojection error E in the form

$$E = \sum_{\alpha=1}^{N} \frac{(\hat{\boldsymbol{\xi}}_\alpha^*, \boldsymbol{u})^2}{(\boldsymbol{u}, V[\hat{\boldsymbol{\xi}}_\alpha]\boldsymbol{u})}. \tag{38}$$

Again, this has the same form as Eq. (11). Hence, it can be minimized by an existing method such as FNS. Let $\hat{\boldsymbol{u}}$ be the solution. From Eqs. (26), (28), (33), and (35), the true value $\bar{\boldsymbol{x}}$ is estimated to be

$$\hat{\hat{\boldsymbol{x}}}_\alpha = \hat{\boldsymbol{x}}_\alpha - \frac{\lambda_\alpha}{2} V[\boldsymbol{x}_\alpha] \Big(\frac{\partial \hat{\boldsymbol{\xi}}}{\partial \boldsymbol{x}}\Big)_\alpha^\top \hat{\boldsymbol{u}} + \tilde{\boldsymbol{x}}_\alpha = \boldsymbol{x}_\alpha - \frac{(\hat{\boldsymbol{\xi}}_\alpha^*, \hat{\boldsymbol{u}}) V[\boldsymbol{x}_\alpha]}{(\hat{\boldsymbol{u}}, V[\hat{\boldsymbol{\xi}}_\alpha]\hat{\boldsymbol{u}})} \Big(\frac{\partial \hat{\boldsymbol{\xi}}}{\partial \boldsymbol{x}}\Big)_\alpha^\top \hat{\boldsymbol{u}}. \tag{39}$$

The resulting $\hat{\hat{\boldsymbol{x}}}_\alpha$ is a better approximation to $\bar{\boldsymbol{x}}_\alpha$ than $\hat{\boldsymbol{x}}_\alpha$ in Eq. (25). Regarding $\hat{\hat{\boldsymbol{x}}}_\alpha$ as $\hat{\boldsymbol{x}}_\alpha$, we repeat the same process until it converges. In the end, $\Delta\hat{\boldsymbol{x}}_\alpha$ in Eq. (29) becomes $\boldsymbol{0}$. This means that *strict ML in the $\boldsymbol{x}$-space coincides with ML in the modified $\hat{\boldsymbol{\xi}}^*$-space*; the mapping from $\boldsymbol{x}$ to $\hat{\boldsymbol{\xi}}^*$ is defined *dynamically* in the course of iterations.

8. EXAMPLES OF STRICT ML

We apply the above procedure to typical examples, where we assume that the x and y coordinates of each point has independent Gaussian noise of mean 0 and variance σ^2. However, the noise variance σ^2 need not be known, because minimization of Eq. (12) is not affected by multiplication of $V[\boldsymbol{x}_\alpha]$ by an arbitrary positive constant. So, we regard σ to be 1 in the computation.

Example 5 (Ellipse fitting). The procedure for fitting the ellipse parameter $\boldsymbol{u}$ in Eq. (4) to a point sequence (x_α, y_α), $\alpha = 1, ..., N$, is given as follows‖, where we remove the scale indeterminacy of $\boldsymbol{u}$ by normalizing it to $\|\boldsymbol{u}\| = 1$:

1. Let $E_0 = \infty$ (a sufficiently large number), $\hat{x}_\alpha = x_\alpha$, $\hat{y}_\alpha = y_\alpha$, and $\tilde{x}_\alpha = \tilde{y}_\alpha = 0$, $\alpha = 1, ..., N$.
2. Computing the following $\boldsymbol{\xi}_\alpha^*$, $\alpha = 1, ..., N$.

$$\boldsymbol{\xi}_\alpha^* = \begin{pmatrix} \hat{x}_\alpha^2 + 2\hat{x}_\alpha\tilde{x}_\alpha \\ 2(\hat{x}_\alpha\hat{y}_\alpha + \hat{y}_\alpha\tilde{x}_\alpha + \hat{x}_\alpha\tilde{y}_\alpha) \\ \hat{y}_\alpha^2 + 2\hat{y}_\alpha\tilde{y}_\alpha \\ 2(\hat{x}_\alpha + \tilde{x}_\alpha) \\ 2(\hat{y}_\alpha + \tilde{y}_\alpha) \\ 1 \end{pmatrix}. \tag{40}$$

3. Let $V[\hat{\boldsymbol{\xi}}_\alpha]$, $\alpha = 1, ..., N$, be the matrices obtained by letting $\sigma = 1$ and replacing x_α and y_α in $V[\boldsymbol{\xi}_\alpha]$ in Eq. (8) by $\hat{x}_\alpha$ and $\hat{y}_\alpha$, respectively.
4. Compute the 6-D unit vector $\boldsymbol{u} = (u_i)$ that minimizes the following function (e.g., by FNS[17]):

$$E(\boldsymbol{u}) = \sum_{\alpha=1}^{N} \frac{(\boldsymbol{u}, \boldsymbol{\xi}_\alpha^*)^2}{(\boldsymbol{u}, V[\hat{\boldsymbol{\xi}}_\alpha]\boldsymbol{u})}. \tag{41}$$

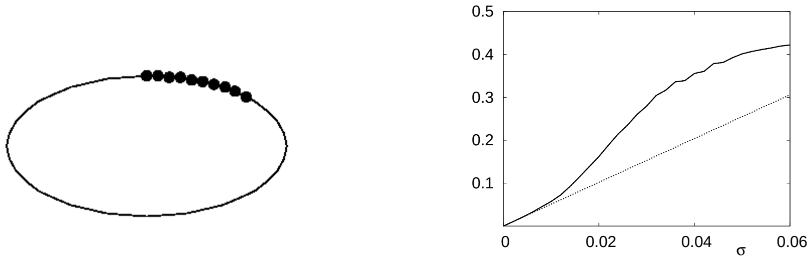
5. Update $\tilde{x}_\alpha$, $\tilde{y}_\alpha$, $\hat{x}_\alpha$, and $\hat{y}_\alpha$ by

$$\begin{pmatrix} \tilde{x}_\alpha \\ \tilde{y}_\alpha \end{pmatrix} \leftarrow \frac{2(\boldsymbol{u}, \boldsymbol{\xi}_\alpha^*)}{(\boldsymbol{u}, V[\hat{\boldsymbol{\xi}}_\alpha]\boldsymbol{u})} \begin{pmatrix} u_1 & u_2 & u_4 \\ u_2 & u_3 & u_5 \end{pmatrix} \begin{pmatrix} \hat{x}_\alpha \\ \hat{y}_\alpha \\ 1 \end{pmatrix}, \qquad \hat{x}_\alpha \leftarrow x_\alpha - \tilde{x}_\alpha, \quad \hat{y}_\alpha \leftarrow y_\alpha - \tilde{y}_\alpha. \tag{42}$$

6. Compute the reprojection error $E = \sum_{\alpha=1}^{N} (\tilde{x}_\alpha^2 + \tilde{y}_\alpha^2)$. If $E \approx E_0$, return $\boldsymbol{u}$ and stop**. Else, let $E_0 \leftarrow E$ and go back to Step 2.

‖The source code is available at: `http://www.iim.ics.tut.ac.jp/~sugaya/public-e.html`

**Alternatively, we can stop when $\boldsymbol{u}$ is sufficiently close to the value in the previous iteration up to sign.



(a) (b)

Figure 2. (a) 10 points on an ellipse. (b) RMS error of the fitted ellipse over 1000 trials vs. the noise level σ (from Kanatani and Sugaya[18,20]). Solid line: ML in the $\boldsymbol{x}$-space. Dashed line: ML in the $\boldsymbol{\xi}$-space (overlapped by the solid line and invisible). Dotted line: KCR lower bound.

The value of $\boldsymbol{u}$ obtained in Step 4 in the first iteration corresponds to the ML solution in the $\boldsymbol{\xi}$-space, minimizing Eq. (11). Its accuracy is numerically examined by Kanatani and Sugaya.[17] The above strict ML scheme was obtained by Kanatani and Sugaya[18,20] by a specific analysis for ellipse fitting. Here, it is derived as a special case of our general theory.

As pointed out by Kanatani and Sugaya,[18,20] however, the resulting accuracy is practically the same as the ML solution in the $\boldsymbol{\xi}$-space; the difference is only in the last few of the significant digits. Figure 2(a) shows 10 points on an ellipse, and Fig. 2(b) shows the RMS error of the fitted ellipse over 1000 trials with independent Gaussian noise of mean 0 and standard deviation σ (pixels) added to the x and y coordinates of each point (from Kanatani and Sugaya[18,20]). The horizontal axis is extended to an unrealistically large value of σ for the sake of comparison, and the solid line plots the result of ML in the $\boldsymbol{x}$-space. The corresponding result of ML in the $\boldsymbol{\xi}$-space is drawn in dashed line but is completely overlapped by the solid line, so it is invisible in the plot. The dotted line shows the theoretical accuracy limit (*KCR lower bound*[3,11,12]).

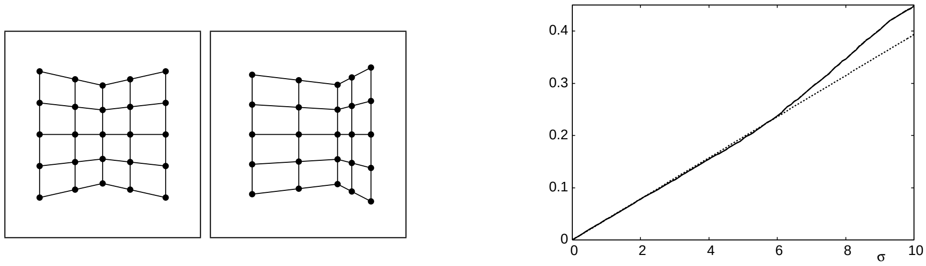
Example 6 (Fundamental matrix computation). The vector $\boldsymbol{u}$ in Eq. (6) that encodes the fundamental matrix $\boldsymbol{F}$ of rank 2 is computed from corresponding points (x_α, y_α) and (x'_α, y'_α), $\alpha = 1, ..., N$, as follows[‖], where we remove the scale indeterminacy of $\boldsymbol{u}$ by normalizing it to $\|\boldsymbol{u}\| = 1$:

1. Let $E_0 = \infty$ (a sufficiently large number), $\hat{x}_\alpha = x_\alpha$, $\hat{y}_\alpha = y_\alpha$, $\hat{x}'_\alpha = x'_\alpha$, $\hat{y}'_\alpha = y'_\alpha$, and $\tilde{x}_\alpha = \tilde{y}_\alpha = \tilde{x}'_\alpha = \tilde{y}'_\alpha = 0$, $\alpha = 1, ..., N$.
2. Compute the following $\boldsymbol{\xi}^*_\alpha$, $\alpha = 1, ..., N$.

$$
\boldsymbol{\xi}^*_\alpha = \begin{pmatrix} \hat{x}_\alpha\hat{x}'_\alpha + \hat{x}'_\alpha\tilde{x}_\alpha + \hat{x}_\alpha\tilde{x}'_\alpha \\ \hat{x}_\alpha\hat{y}'_\alpha + \hat{y}'_\alpha\tilde{x}_\alpha + \hat{x}_\alpha\tilde{y}'_\alpha \\ \hat{x}_\alpha + \tilde{x}_\alpha \\ \hat{y}_\alpha\hat{x}'_\alpha + \hat{x}'_\alpha\tilde{y}_\alpha + \hat{y}_\alpha\tilde{x}'_\alpha \\ \hat{y}_\alpha\hat{y}'_\alpha + \hat{y}'_\alpha\tilde{y}_\alpha + \hat{y}_\alpha\tilde{y}'_\alpha \\ \hat{y}_\alpha + \tilde{y}_\alpha \\ \hat{x}'_\alpha + \tilde{x}'_\alpha \\ \hat{y}'_\alpha + \tilde{y}'_\alpha \\ 1 \end{pmatrix}. \tag{43}
$$

3. Let $V[\hat{\boldsymbol{\xi}}_\alpha]$, $\alpha = 1, ..., N$, be the matrices obtained by letting $\sigma = 1$ and replacing x_α, y_α, x'_α, and y'_α in $V[\boldsymbol{\xi}_\alpha]$ in Eq. (9) by $\hat{x}_\alpha$, $\hat{y}_\alpha$, $\hat{x}'_\alpha$, and $\hat{y}'_\alpha$, respectively.
4. Compute the 9-D unit vector $\boldsymbol{u} = (u_i)$ that minimizes the following function subject to the constraint that the resulting fundamental matrix $\boldsymbol{F}$ has rank 2 (e.g., by EFNS[16]):

$$
E(\boldsymbol{u}) = \sum_{\alpha=1}^{N} \frac{(\boldsymbol{u}, \boldsymbol{\xi}^*_\alpha)^2}{(\boldsymbol{u}, V[\hat{\boldsymbol{\xi}}_\alpha]\boldsymbol{u})}. \tag{44}
$$



(a) (b)

Figure 3. (a) Planer grid patterns viewed from two angles. (b) RMS error of the fitted fundamental matrix over 1000 trials vs. the noise level σ (from Kanatani and Sugaya[19,21]). Solid line: ML in the $\boldsymbol{x}$-space. Dashed line: ML in the $\boldsymbol{\xi}$-space (overlapped by the solid line and invisible). Dotted line: KCR lower bound.

5. Update $\tilde{x}_\alpha$, $\tilde{y}_\alpha$, $\tilde{x}'_\alpha$, and $\tilde{y}'_\alpha$ by

$$\begin{pmatrix} \tilde{x}_\alpha \\ \tilde{y}_\alpha \end{pmatrix} \leftarrow \frac{(\boldsymbol{u}, \boldsymbol{\xi}^*_\alpha)}{(\boldsymbol{u}, V[\hat{\boldsymbol{\xi}}_\alpha]\boldsymbol{u})} \begin{pmatrix} u_1 & u_2 & u_3 \\ u_4 & u_5 & u_6 \end{pmatrix} \begin{pmatrix} \hat{x}'_\alpha \\ \hat{y}'_\alpha \\ 1 \end{pmatrix}, \qquad \begin{pmatrix} \tilde{x}'_\alpha \\ \tilde{y}'_\alpha \end{pmatrix} \leftarrow \frac{(\boldsymbol{u}, \boldsymbol{\xi}^*_\alpha)}{(\boldsymbol{u}, V[\hat{\boldsymbol{\xi}}_\alpha]\boldsymbol{u})} \begin{pmatrix} u_1 & u_4 & u_7 \\ u_2 & u_5 & u_8 \end{pmatrix} \begin{pmatrix} \hat{x}_\alpha \\ \hat{y}_\alpha \\ 1 \end{pmatrix}. \tag{45}$$

6. Update $\hat{x}_\alpha$, $\hat{y}_\alpha$, $\hat{x}'_\alpha$, and $\hat{y}'_\alpha$ by

$$\hat{x}_\alpha \leftarrow x_\alpha - \tilde{x}_\alpha, \qquad \hat{y}_\alpha \leftarrow y_\alpha - \tilde{y}_\alpha, \qquad \hat{x}'_\alpha \leftarrow x'_\alpha - \tilde{x}'_\alpha, \qquad \hat{y}'_\alpha \leftarrow y'_\alpha - \tilde{y}'_\alpha. \tag{46}$$

7. Compute the reprojection error $E = \sum_{\alpha=1}^N (\tilde{x}_\alpha^2 + \tilde{y}_\alpha^2 + \tilde{x}'^2_\alpha + \tilde{y}'^2_\alpha)$. If $E \approx E_0$, return $\boldsymbol{u}$ and stop[††]. Else, let $E_0 \leftarrow E$ and go back to Step 2.

As in the ellipse fitting case, the value of $\boldsymbol{u}$ obtained in Step 4 in the first iteration corresponds to the ML solution in the $\boldsymbol{\xi}$-space, minimizing Eq. (11). Its accuracy is numerically examined by Kanatani and Sugaya.[15] The above strict ML scheme was obtained by Kanatani and Sugaya[19,21] by a specific analysis for fundamental matrix computation. Here, it is derived as a special case of our general theory.

As pointed out by Kanatani and Sugaya,[19,21] however, the resulting accuracy is practically the same as the ML solution in the $\boldsymbol{\xi}$-space; the difference is only in the last few of the significant digits. Figure 3(a) shows two planar grid patterns viewed from two angles, and Fig. 3(b) shows the RMS error of the computed fundamental matrix over 1000 trials with independent Gaussian noise of mean 0 and standard deviation σ (pixels) added to the x and y coordinates of each grid point (from Kanatani and Sugaya[19,21]). The horizontal axis is extended to an unrealistically large value of σ for the sake of comparison, and the solid line plots the result of ML in the $\boldsymbol{x}$-space. The corresponding result of ML in the $\boldsymbol{\xi}$-space is drawn in dashed line but is completely overlapped by the solid line, so it is invisible in the plot. The dotted line shows the KCR lower bound.

9. APPLICATION TO OPTIMAL CORRECTION

Our strict ML computation encompasses *optimal correction*:[11] we optimally correct the datum $\boldsymbol{x}$ so as to satisfy the constraint

$$F(\boldsymbol{x}; \boldsymbol{u}) = 0, \tag{47}$$

where the parameter $\boldsymbol{u}$ is given and fixed. Rewriting Eq. (43) in the form of Eq. (2), we state the problem as minimization of the reprojection error

$$E = (\boldsymbol{x} - \bar{\boldsymbol{x}}, V[\boldsymbol{x}]^{-1}(\boldsymbol{x} - \bar{\boldsymbol{x}})) \quad \text{subject to the constraint} \quad (\boldsymbol{\xi}(\bar{\boldsymbol{x}}), \boldsymbol{u}) = 0. \tag{48}$$

Thus, the optimal correction procedure is obtained by simply removing the computation of $\boldsymbol{u}$ in the procedure described in Sect. 7.

Example 7 (Perpendicular to an ellipse). Given a point (x, y) and an ellipse in the form of Eq. (3), we want to compute the foot $(\hat{x}, \hat{y})$ of the perpendicular from (x, y) (Fig. 4(a)). It is computed as follows[‖] ($\boldsymbol{u}$ is defined by Eq. (4)):

[††]Alternatively, we can stop when $\boldsymbol{u}$ is sufficiently close to the value in the previous iteration up to sign.

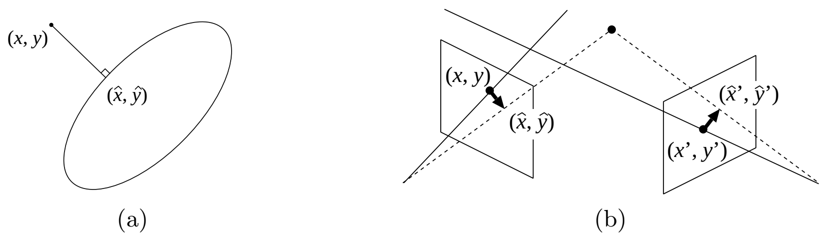


Figure 4. (a) Drawing a perpendicular to an ellipse. (b) Computing the 3-D position from noisy correspondence pair.

1. Let $E_0 = \infty$ (a sufficiently large number), $\hat{x} = x$, $\hat{y} = y$, and $\tilde{x} = \tilde{y} = 0$.
2. Compute the following $\boldsymbol{\xi}^*$:

$$\boldsymbol{\xi}^* = \begin{pmatrix} \hat{x}^2 + 2\hat{x}\tilde{x} \\ 2(\hat{x}\hat{y} + \hat{y}\tilde{x} + \hat{x}\tilde{y}) \\ \hat{y}^2 + 2\hat{y}\tilde{y} \\ 2(\hat{x} + \tilde{x}) \\ 2(\hat{y} + \tilde{y}) \\ 1 \end{pmatrix}. \tag{49}$$

3. Let $V[\hat{\boldsymbol{\xi}}]$ be the matrix obtained by letting $\sigma = 1$ and replacing x_α and y_α in $V[\boldsymbol{\xi}_\alpha]$ in Eq. (8) by $\hat{x}$ and $\hat{y}$, respectively.
4. Update $\tilde{x}$, $\tilde{y}$, $\hat{x}$, and $\hat{y}$ by

$$\begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} \leftarrow \frac{2(\boldsymbol{u}, \boldsymbol{\xi}^*)}{(\boldsymbol{u}, V[\hat{\boldsymbol{\xi}}]\boldsymbol{u})} \begin{pmatrix} u_1 & u_2 & u_4 \\ u_2 & u_3 & u_5 \end{pmatrix} \begin{pmatrix} \hat{x} \\ \hat{y} \\ 1 \end{pmatrix}, \qquad \hat{x} \leftarrow x - \tilde{x}, \qquad \hat{y} \leftarrow y - \tilde{y}. \tag{50}$$

5. Compute the reprojection error $E = \tilde{x}^2 + \tilde{y}^2$. If $E \approx E_0$, return $(\hat{x}, \hat{y})$ and stop. Else, let $E_0 \leftarrow E$ and go back to Step 2.

As is well known, the perpendicular to an ellipse can be obtained by solving simultaneous algebraic equations. It seems, however, the the above simple procedure has not been known. Usually, the computation converges after 3 or 4 iterations, but even the first solution has sufficient accuracy for practical use.

Example 8 (Triangulation). When the fundamental matrix $\boldsymbol{F}$ is known and a noisy correspondence pair (x, y) and (x', y') is given, we can optimally reconstruct its 3-D position by minimally correcting (x, y) and (x', y') so as to satisfy the epipolar equation[‡‡] in Eq. (5) determined by $\boldsymbol{F}$ (Fig. 4(b)). The optimally corrected positions $(\hat{x}, \hat{y})$ and $(\hat{x}', \hat{y}')$ that minimize the sum of square distances from (x, y) and (x', y') are computed as follows[‖] ($\boldsymbol{u}$ is defined by Eq. (6)):

1. Let $E_0 = \infty$ (a sufficiently large number), $\hat{x} = x$, $\hat{y} = y$, $\hat{x}' = x'$, and $\hat{y}' = y'$, $\tilde{x} = \tilde{y} = \tilde{x}' = \tilde{y}' = 0$.
2. Compute the following $\boldsymbol{\xi}^*$:

$$\boldsymbol{\xi}^* = \begin{pmatrix} \hat{x}\hat{x}' + \hat{x}'\tilde{x} + \hat{x}\tilde{x}' \\ \hat{x}\hat{y}' + \hat{y}'\tilde{x} + \hat{x}\tilde{y}' \\ \hat{x} + \tilde{x} \\ \hat{y}\hat{x}' + \hat{x}'\tilde{y} + \hat{y}\tilde{x}' \\ \hat{y}\hat{y}' + \hat{y}'\tilde{y} + \hat{y}\tilde{y}' \\ \hat{y} + \tilde{y} \\ \hat{x}' + \tilde{x}' \\ \hat{y}' + \tilde{y}' \\ 1 \end{pmatrix}. \tag{51}$$

3. Let $V[\hat{\boldsymbol{\xi}}]$ be the matrix obtained by letting $\sigma = 1$ and replacing x_α, y_α, x'_α, and y'_α in $V[\boldsymbol{\xi}_\alpha]$ in Eq. (9) by $\hat{x}$, $\hat{y}$, $\hat{x}'$, and $\hat{y}'$, respectively.

[‡‡]The lines of sight determined by points (x, y) and (x', y') intersect in the scene if and only if the epipolar equation in Eq. (5) holds.[10]

4. Update $\tilde{x}$, $\tilde{y}$, $\tilde{x}'$, and $\tilde{y}'$ by

$$\begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} \leftarrow \frac{(\boldsymbol{u}, \boldsymbol{\xi}^*)}{(\boldsymbol{u}, V[\hat{\boldsymbol{\xi}}]\boldsymbol{u})} \begin{pmatrix} u_1 & u_2 & u_3 \\ u_4 & u_5 & u_6 \end{pmatrix} \begin{pmatrix} \hat{x}' \\ \hat{y}' \\ 1 \end{pmatrix}, \qquad \begin{pmatrix} \tilde{x}' \\ \tilde{y}' \end{pmatrix} \leftarrow \frac{(\boldsymbol{u}, \boldsymbol{\xi}^*)}{(\boldsymbol{u}, V[\hat{\boldsymbol{\xi}}]\boldsymbol{u})} \begin{pmatrix} u_1 & u_4 & u_7 \\ u_2 & u_5 & u_8 \end{pmatrix} \begin{pmatrix} \hat{x} \\ \hat{y} \\ 1 \end{pmatrix}. \tag{52}$$

5. Update $\hat{x}$, $\hat{y}$, $\hat{x}'$, and $\hat{y}'$ by

$$\hat{x} \leftarrow x - \tilde{x}, \qquad \hat{y} \leftarrow y - \tilde{y}, \qquad \hat{x}' \leftarrow x' - \tilde{x}', \qquad \hat{y}' \leftarrow y' - \tilde{y}'. \tag{53}$$

6. Compute the reprojection error $E = \tilde{x}^2 + \tilde{y}^2 + \tilde{x}'^2 + \tilde{y}'^2$. If $E \approx E_0$, return $(\hat{x}, \hat{y})$ and $(\hat{x}', \hat{y}')$ and stop. Else, let $E_0 \leftarrow E$ and go back to Step 2.

This procedure is nothing but the optimal stereo triangulation of Kanatani et al.[22] A popular method for optimal triangulation is due to Hartley and Sturm,[9] who determined the epipolar lines of the corresponding points by algebraically solving a 6-degree polynomial. Their method is widely regarded as a standard tool for triangulation. Kanatani et al.[22] experimentally confirmed that their solution is identical to that of Hartley and Sturm[9] yet the computation is significantly faster.

10. CONCLUDING REMARKS

This paper has presented a unified numerical scheme for strict ML computation for geometric fitting problems. While methods assuming Gaussian noise in the transformed data space ($\boldsymbol{\xi}$-space) have intensively been studied in the past, we assume Gaussian noise in the original data space ($\boldsymbol{x}$-space). We have shown that strict ML in the $\boldsymbol{x}$-space reduces to iterations of ML in the *dynamically defined* $\boldsymbol{\xi}^*$-space. The computation is done in the low dimensional space of the unknown $\boldsymbol{u}$, and the true values $\bar{\boldsymbol{x}}_\alpha$ of the data $\boldsymbol{x}_\alpha$ are also ML estimated.

The strict ML schemes have already been derived specifically for ellipse fitting[18,20] and fundamental matrix computation.[19,21] They are regarded as special cases of our general theory. We have also shown that our theory encompasses optimal correction problems: compact schemes for computing perpendiculars to an ellipse and optimally triangulating stereo images are obtained as special cases. Thus, our general theory provides a unified problem-independent point of view. At the moment, however, it is difficult to say whether or not our unified method is more efficient than problem-dependent computation in a high dimensional parameter space, since the efficiency of the latter heavily depends on how the particularities of the problem are exploited in the implementation.

As experimentally pointed out by Kanatani and Sugaya,[18–21] however, ML in the $\boldsymbol{x}$-space practically coincides with ML in the $\boldsymbol{\xi}$-space. This means that as long as the covariance is correctly evaluated, regarding not strictly Gaussian noise in the $\boldsymbol{\xi}$-space as Gaussian practically does not affect the final solution. In the present stage, the role of our theory is limited to comparing the accuracy of existing methods and evaluating the effect of various approximations involved. In the future, however, various new applications are expected based on our theory.

Acknowledgments. The author thanks Yasuyuki Sugaya of Toyohashi University of Technology, Japan, Mike Brooks and Wojciech Chojnacki of the University Adelaide, Australia, Nikolai Chernov of the University of Alabama at Birmingham, U.S.A., and Wolfgang Förstner of the University of Bonn, for helpful discussions.

REFERENCES

[1] A. Bartoli and P. Sturm, Nonlinear estimation of fundamental matrix with minimal parameters, *IEEE Trans. Patt. Anal. Mach. Intell.*, **26**-3 (2004-3), 426–432.

[2] A. Björck, *Numerical Methods for Least Squares Problems*, SIAM, Philadelphia, PA, U.S.A., 1996.

[3] N. Chernov and C. Lesort, Statistical efficiency of curve fitting algorithms, *Comput. Stat. Data Anal.*, **47**-4 (2004-11), 713–728.

[4] W. Chojnacki, M. J. Brooks, A. van den Hengel, and D. Gawley, On the fitting of surfaces to data with covariances, *IEEE Trans. Patt. Anal. Mach. Intell.*, **22**-11 (2000-11), 1294–1303.

[5] W. Chojnacki, M. J. Brooks, A. van den Hengel, and D. Gawley, A new constrained parameter estimator for computer vision applications, *Image Vision Comput.*, **22**-2 (2004-2), 85–91.

[6] W. Föstner, On weighting and choosing constraints for optimally reconstructing the geometry of image triplets, *Proc. 6th Euro. Conf. Comput. Vision*, June/July 2000, Dublin, Ireland, Vol. 2, pp. 669–701.

[7] W. Gander, H. Golub, and R. Strebel, Least-squares fitting of circles and ellipses, *BIT*, **34**-4 (1994-12), 558–578.

[8] M. Harker and P. O'Leary, First order geometric distance (The myth of Sampsonus), *Proc. 17th Brit. Mach. Vision Conf.*, September 2006, Edinburgh, U.K., Vol. 1, pp. 87–96.

[9] R. I. Hartley and P. Sturm, Triangulation, *Comput. Vision Image Understand.*, **68**-2 (1997-11), 146–157.

[10] R. Hartley and A. Zisserman, *Multiple View Geometry in Computer Vision*, Cambridge University Press, Cambridge, U.K., 2000.

[11] K. Kanatani, *Statistical Optimization for Geometric Computation: Theory and Practice*, Elsevier Science, Amsterdam, The Netherlands, 1996; reprinted Dover, New York, U.S.A., 2005.

[12] K. Kanatani, Cramer-Rao lower bounds for curve fitting, *Graphical Models Image Process.*, **60**-2 (1998-3), 93–99.

[13] K. Kanatani, Statistical optimization for geometric fitting: Theoretical accuracy analysis and high order error analysis, *Int. J. Compt. Vision*, **80**-2 (2008-11), 167–188.

[14] K. Kanatani and N. Ohta, Comparing optimal three-dimensional reconstruction for finite motion and optical flow, *J. Electr. Imaging*, **12**-3 (2003-7), 478–488.

[15] K. Kanatani and Y. Sugaya, High accuracy fundamental matrix computation and its performance evaluation, *IEICE Trans. Inf. Syst.*, **E90-D**-2 (2007-2), 579–585.

[16] K. Kanatani and Y. Sugaya, Extended FNS for constrained parameter estimation, *Proc. Meeting Image Recognition and Understanding*, July–August 2007, Hiroshima, Japan, pp. 219–226.

[17] K. Kanatani and Y. Sugaya, Performance evaluation of iterative geometric fitting algorithms, *Comput. Stat. Data Anal.*, **52**-2 (2007-10), 1208–1222.

[18] K. Kanatani and Y. Sugaya, Compact algorithm for truly optimal ellipse fitting, *Proc. 10th Symp. Sensing via Imaging Information*, June 2008, Yokohama, Japan, pp. IN1-12-1–IN1-12-7.

[19] K. Kanatani and Y. Sugaya, Small algorithm for fundamental matrix computation, *Proc. Meeting on Image Recognition and Understanding*, July 2008, Karuizawa, Japan, pp. 947–954.

[20] K. Kanatani and Y. Sugaya, Compact algorithm for strictly ML ellipse fitting, *Proc. 19th Int. Conf. Patt. Recog.*, December 2008, Tampa, FL, U.S.A., to appear.

[21] K. Kanatani and Y. Sugaya, Compact fundamental matrix computation, *Proc. 3rd Pacific Rim Symp. Image and Video Technology*, January 2009, Tokyo, Japan, to appear.

[22] K. Kanatani, Y. Sugaya, and H. Niitsuma, Triangulation from two views revisited: Hartley-Sturm vs. optimal correction, *Proc. 19th British Machine Vision Conf.*, September 2008, Leeds, U.K., pp. 173–182.

[23] Y. Leedan and P. Meer, Heteroscedastic regression in computer vision: Problems with bilinear constraint, *Int. J. Comput. Vision*, **37**-2 (2000-6), 127–150.

[24] E. M. Mikhail and F. Ackermann, *Observations and Least Squares*, University Press of America, Lanham, MD., U.S.A., 1976.

[25] P. Sturm and P. Gargallo, Conic fitting using the geometric distance, *Proc. 8th Asian Conf. Comput. Vision (ACCV2007)*, November 2007, Tokyo, Japan, Vol. 2, pp 784–795.

[26] B. Triggs, P. F. McLauchlan, R. I. Hartley, and A. Fitzgibbon, Bundle adjustment—A modern synthesis, in B. Triggs, A. Zisserman, and R. Szeliski (Eds.), *Vision Algorithms: Theory and Practice*, Springer, Berlin, Germany, 2000.